

A MULTI-FIDELITY STOCHASTIC COLLOCATION METHOD FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH RANDOM INPUT DATA

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Over the last few years there have been dramatic advances in the area of uncertainty quantification. In particular, we have seen a surge of interest in developing efficient, scalable, stable, and convergent computational methods for solving differential equations with random inputs. Stochastic collocation (SC) methods, which inherit both the ease of implementation of sampling methods like Monte Carlo and the robustness of nonsampling ones like stochastic Galerkin to a great deal, have proved extremely useful in dealing with differential equations driven by random inputs. In this work we propose a novel enhancement to stochastic collocation methods using deterministic model reduction techniques. Linear parabolic partial differential equations with random forcing terms are analysed. The input data are assumed to be represented by a finite number of random variables. A rigorous convergence analysis, supported by numerical results, shows that the proposed technique is not only reliable and robust but also efficient.

KEY WORDS: *collocation, stochastic partial differential equations, sparse grid, smolyak algorithm, finite element, proper orthogonal decomposition, multifidelity*

1. INTRODUCTION

The effectiveness of stochastic partial differential equations (SPDEs) in modeling complex systems is a well-known fact. One can name wave propagation [1], diffusion through heterogeneous random media [2], randomly forced Burgers and Navier-Stokes equations (see e.g., [3–6] and the references therein) as a couple of examples. Currently, Monte Carlo is one of the most widely used tools in simulating models driven by SPDEs. However, Monte Carlo simulations are generally very expensive. To meet this concern, methods based on the Fourier analysis with respect to the Gaussian (rather than the Lebesgue) measure, have been investigated in recent decades. More specifically, the Cameron–Martin version of the Wiener Chaos expansion (see, e.g., [7, 8] and the references therein) is among the earlier efforts. Sometimes, the Wiener Chaos expansion (WCE for short) is also referred to as the Hermite polynomial chaos expansion. The term polynomial chaos was coined by Nobert Wiener [9]. In Wiener's work, Hermite polynomials served as an orthogonal basis. The validity of the approach was then proved in [7]. There is a long history of using WCE as well as other polynomial chaos expansions in problems in physics and engineering. See, e.g., [10–13], etc. Applications of the polynomial chaos to stochastic PDEs considered in the literature typically deal with stochastic input generated by a finite number of random variables (see, e.g. [14–17]). This assumption is usually introduced either directly or via a representation of the stochastic input by a truncated Karhunen–Loève (KL) expansion. Stochastic finite element methods based on the Karhunen–Loève expansion and Hermite polynomial chaos expansion [14, 15] have been developed by Ghanem and other authors. Karniadakis et al. generalized this idea to other types of randomness and polynomials [16, 18, 19]. The stochastic finite element procedure often results in a set of coupled deterministic equations which requires additional effort to be solved. To resolve this issue, the stochastic collocation (SC) method was introduced.

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In this method one repeatedly executes an established deterministic code on a prescribed node in the random space defined by the random inputs. The idea can be found in early works such as [20, 21]. In these works mostly tensor products of one-dimensional nodes (e.g., Gauss quadrature) are employed. Tensor product construction despite making mathematical analysis more accessible (cf. [22]) leads to the curse of dimensionality since the total number of nodes grows exponentially fast as the number of random parameters increases. In recent years we are experiencing a surge of interest in the high-order stochastic collocation approach following [23]. The use of sparse grids from multivariate interpolation analysis is a distinct feature of the work in [23]. A sparse grid, being a subset of the full tensor grid, can retain many of the accuracy properties of the tensor grid. While keeping high-order accuracy, it can significantly reduce the number of nodes in higher random dimensions. Further reduction in the number of nodes was pursued in [24–27]. Applications of stochastic Galerkin and SC methods take a wide range. Here we mention some of the more representative works. It includes Burgers equation [28, 29], fluid dynamics [16, 30–33], flow-structure interactions [34], hyperbolic problems [35–37], model construction and reduction [38–40], random domains with rough boundaries [41–44], etc.

Along with an attempt to reduce the number of nodes used by sparse grid stochastic collocation, one can try to employ more efficient deterministic algorithms. The current trend is to repeatedly execute a full-scale underlying deterministic simulation on prescribed nodes in the random space. However, model reduction techniques can be employed to create a computationally cheap deterministic algorithm that can be used for most of the grid points. This way we can limit the employment of an established while computationally expensive algorithm to only a relatively small number of points. A related method is being used by Willcox and her team but in the context of optimization [45]. “Multifidelity,” which we also adopt, is the term they employed in their work. Reduced order modeling, using proper orthogonal decompositions (POD) along with Galerkin projection, for fluid flows has seen extensive applications studied in [46–55]. Proper orthogonal decomposition (POD) was introduced in Pearson [56] and Hotelling [57]. Since the work of Pearson and Hotelling, many have studied or used POD in a range of fields such as oceanography [58], fluid mechanics [46, 48], system feedback control [59–64], and system modeling [49, 52, 54, 65]. In this work we analyze linear parabolic partial differential equations with random forcing terms. We propose a novel method which dramatically decreases the computational cost. The idea of the method is very simple. For each point of the stochastic parameter domain we search to see if the resulting deterministic problem is already solved for a sufficiently close problem. If yes, we use the solution to the nearby problem to create POD basis functions and we employ the POD-Galerkin method to solve the original problem. We provide a rigorous convergence analysis for our proposed method. Finally, it is shown by numerical examples that the results of numerical computation are consistent with theoretical conclusions.

2. PROBLEM DEFINITION

Let $D \subset \mathbb{R}^2$ be a bounded, connected, and polygonal domain and (Ω, \mathcal{F}, P) denote a complete probability space with sample space Ω , which corresponds to the set of all possible outcomes. \mathcal{F} is the σ algebra of events, and $P : \mathcal{F} \rightarrow [0, 1]$ is the probability measure. In this section, we consider the stochastic linear parabolic initial-boundary value problem: find a random field $u : [0, T] \times \overline{D} \times \Omega \rightarrow \mathbb{R}$, such that P -almost surely the following equations hold

$$\begin{aligned} \partial_t u(t, \mathbf{x}, \omega) - \Delta u(t, \mathbf{x}, \omega) &= f(t, \mathbf{x}, \omega) & \text{in } & (0, T] \times D \times \Omega, \\ u(t, \mathbf{x}, \omega) &= 0 & \text{on } & (0, T] \times \partial D \times \Omega, \\ u(0, \mathbf{x}, \omega) &= 0 & \text{on } & D \times \Omega. \end{aligned} \quad (1)$$

Existence and uniqueness of the solution of (1), as stated in [66], can be achieved by assuming that the random forcing field $f : [0, T] \times \overline{D} \times \Omega \ni (t, \mathbf{x}, \omega) \mapsto f(t, \mathbf{x}, \omega) \in \mathbb{R}$ satisfies

$$\int_0^T \int_D f^2(t, \mathbf{x}, \omega) dx dt < +\infty, \quad P\text{-a.e. in } \Omega. \quad (2)$$

Following [22] and inspired by the truncated KL expansion [67], we make the assumption that the random field f depends on a finite number of independent random variables. More specifically,

$$f(t, \mathbf{x}, \omega) = f(t, \mathbf{x}, \mathbf{y}(\omega)) \quad \text{on} \quad [0, T] \times \bar{D} \times \Omega, \quad (3)$$

where $\mathbf{y}(\omega) = (y_1(\omega), \dots, y_r(\omega))$ and $r \in \mathbb{N}_+$. Let $\Gamma_n = y_n(\Omega)$ denote the image of the random variable y_n , for $n = 1, \dots, r$, and $\Gamma = \prod_{n=1}^r \Gamma_n$. Furthermore, similar to [27], we make the assumption that the random variables $\mathbf{y} = (y_1, \dots, y_r)$ have $\rho : \Gamma \rightarrow \mathbb{R}_+$ as their joint probability density function. Define $L_\rho^2(\Gamma)$ and V_ρ to be given by

$$L_\rho^2(\Gamma) := \{\mathbf{y} \in \Gamma : \int_\Gamma \|\mathbf{y}\|^2 \rho d\mathbf{y} < \infty\},$$

and

$$V_\rho = L^2(0, T; H_0^1(D)) \otimes L_\rho^2(\Gamma),$$

with inner product

$$(u, v)_{V_\rho} = \int_\Gamma (u(\mathbf{y}), v(\mathbf{y}))_{L^2(0, T; H_0^1(D))} \rho d\mathbf{y},$$

where

$$(u(\mathbf{y}), v(\mathbf{y}))_{L^2(0, T; H_0^1(D))} = \int_0^T \int_D \nabla u(t, \mathbf{x}, \mathbf{y}) \cdot \nabla v(t, \mathbf{x}, \mathbf{y}) dx dt.$$

A function $u \in V_\rho$ is called a *weak solution* (see e.g., [66]) of problem (1) if

$$\int_\Gamma \int_D \partial_t u v dx \rho d\mathbf{y} + \int_\Gamma \int_D \nabla u \cdot \nabla v dx \rho d\mathbf{y} = \int_\Gamma \int_D f v dx \rho d\mathbf{y}, \quad \forall v \in H_0^1(D) \otimes L_\rho^2(\Gamma) \text{ and } \forall t \in (0, T], \quad (4)$$

and $u(0, \mathbf{x}, \mathbf{y}) = 0$, ρ -almost everywhere in Γ . The existence and uniqueness of the solution of problem (4) is a direct consequence of assumption (2) on f ; see [68].

For each fixed $t \in (0, T]$, the solution u to (4) can be viewed as a mapping $u : \Gamma \rightarrow H_0^1(D)$. In order to emphasize the dependence on the variable \mathbf{y} , we use the notations $u(\mathbf{y})$ and $f(\mathbf{y})$. As in [66], problem (4) can be equivalently expressed as finding $u(\mathbf{y}) \in H_0^1(D)$ such that ρ -almost everywhere in Γ , $u(0, \mathbf{x}, \mathbf{y}) = 0$, and

$$\int_D \partial_t u(\mathbf{y}) v dx + \int_D \nabla u(\mathbf{y}) \cdot \nabla v dx = \int_D f(\mathbf{y}) v dx, \quad \forall v \in H_0^1(D) \text{ and } \forall t \in (0, T], \quad \rho\text{-a.e. in } \Gamma. \quad (5)$$

3. MULTIFIDELITY COLLOCATION METHOD

In this section we explain our proposed multifidelity stochastic collocation method while applying it to the weak form (5). We are in fact seeking a numerical approximation to the exact solution of (5) in a finite dimensional subspace $V_{\rho, h}$ of V_ρ given by $V_{\rho, h} = L^2(0, T; H_h(D)) \otimes \mathcal{P}_{\mathbf{p}}(\Gamma)$, where $H_h(D) \subset H_0^1(D)$ is a standard finite element space and $\mathcal{P}_{\mathbf{p}}(\Gamma) \subset L_\rho^2(\Gamma)$ is the span of tensor product polynomials with degree at most $\mathbf{p} = (p_1, \dots, p_r)$. Choose $\eta > 0$ to be a small real number. The procedure for solving (5) is divided into two parts:

1. Fix $\mathbf{y} \in \Gamma$, and search the η neighborhood $B_\eta(\mathbf{y}) \subset \Gamma$ of \mathbf{y} . If problem (5) is not already solved for any nearby problem with $\mathbf{y}' \in B_\eta(\mathbf{y})$, solve problem (5) using a regular backward Euler finite element method at \mathbf{y} and let $\mathbf{y}' = \mathbf{y}$. In contrast, if Eq. (5) is already solved for some points in $B_\eta(\mathbf{y})$, choose the closest one to \mathbf{y} and call it \mathbf{y}' . In either case, use the solution at $\mathbf{y}' \in B_\eta(\mathbf{y})$ to find a small number $d \in \mathbb{N}_+$ of suitable orthonormal basis functions $\{\psi_j(\mathbf{y}')\}_{j=1}^d \subset H_h(D)$ using the POD method. Now use Galerkin projection on to the subspace $X^d(\mathbf{y}') = \text{span}\{\psi_j(\mathbf{y}')\}_{j=1}^d$ to find

$$\{u_d^m(\mathbf{y})\}_{m=1}^N \subset X^d(\mathbf{y}') \subset H_h(D),$$

such that

$$(u_d^m, v_d) + k(\nabla u_d^m, \nabla v_d) = k(f^m(\mathbf{y}), v_d) + (u_d^{m-1}, v_d), \quad \forall v_d \in X^d(\mathbf{y}'), \quad m = 1, \dots, N, \quad (6)$$

and $u_d^0 = 0$, where $N \in \mathbb{N}_+$ is the number of time steps, and $k = T/N$ denotes the time step increments. It is worth mentioning that (\cdot, \cdot) denotes the L^2 inner product. Note that we are employing a backward Euler scheme to discretize time.

2. Collocate (6) on zeros of suitable orthogonal polynomials and build the interpolated discrete solution

$$\{u_{d,\mathbf{p}}^m\}_{m=1}^N \subset H_h(D) \otimes \mathcal{P}_{\mathbf{p}}(\Gamma), \quad (7)$$

using

$$u_{d,\mathbf{p}}^m(x, \mathbf{y}) = \mathcal{I}_{\mathbf{p}} u_d^m(x, \mathbf{y}) = \sum_{j_1=1}^{p_1+1} \cdots \sum_{j_r=1}^{p_r+1} u_d^m(x, y_{j_1}, \dots, y_{j_r}) (l_{j_1}(\mathbf{y}) \otimes \cdots \otimes l_{j_r}(\mathbf{y})), \quad m = 1, \dots, N, \quad (8)$$

where the functions $\{l_{j_k}\}_{k=1}^r$ can be taken as Lagrange polynomials. Using this formula, as described in [22], mean value and variance of u can also be easily approximated.

4. POD

In this section, we choose a fixed $\mathbf{y}' \in B_{\eta}(\mathbf{y}) \subset \Gamma$ and drop the dependence of Eq. (5) on \mathbf{y}' , for notational conveniences. Therefore, we consider the problem of finding $w \in H_0^1(D)$ such that

$$(w_t, v) + (\nabla w, \nabla v) = (g, v), \quad \forall v \in H_0^1(D), \quad (9)$$

and $w(\mathbf{x}, 0) = 0$, for all $\mathbf{x} \in D$. Note that $g = f(\mathbf{y}')$. Let $t_m = mk, k = 0, \dots, N$, where k denotes the time step increments. Assume \mathfrak{T}_h to be a uniformly regular family of triangulation of \bar{D} (see [69, 70]). The finite element space is taken as

$$H_h(D) = \{v_h \in H_0^1(D) \cap C^0(D) : v_h|_K \in P_s(K), \quad \forall K \in \mathfrak{T}_h\},$$

where $s \in \mathbb{N}_+$ and $P_s(K)$ is the space of polynomials of degree $\leq s$ on K . Write $w^m(\mathbf{x}) = w(\mathbf{x}, t_m)$, and let w_h^m denote the fully discrete approximation of w resulting from solving the problem of finding $w_h^m \in H_h(D)$ such that $w_h^0(\mathbf{x}) = 0$ and for $m = 1, \dots, N$,

$$(w_h^m, v_h) + k(\nabla w_h^m, \nabla v_h) = k(g^m, v_h) + (w_h^{m-1}, v_h), \quad \forall v_h \in H_h(D), \quad m = 1, \dots, N. \quad (10)$$

It is easy to prove that problem (10) has a unique solution $w_h^m \in H_h(D)$, provided that $g^m \in L^2(D)$ (see [69]). One can also show that if $w_t \in H^{s+1}(D)$ and $w_{tt} \in L^2(D)$, the following error estimates hold:

$$\|w^m - w_h^m\|_0 \leq Ch^{s+1} \int_0^{t_m} \|w_t\|_{s+1} dt + Ck \int_0^{t_m} \|w_{tt}\|_0 dt, \quad m = 1, \dots, N, \quad (11)$$

where $\|\cdot\|_s$ denotes the $H^s(D)$ norm and C indicates a positive constant independent of the spatial and temporal mesh sizes, possibly different at distinct occurrences.

For the so-called snapshots $U_i := w_h^{m_i} \in H_h(D)$, $i = 1, \dots, \ell$, where $1 \leq m_1 < m_2 < \cdots < m_\ell \leq N$, let

$$\mathcal{V} = \text{span}\{U_1, \dots, U_\ell\}.$$

Assume at least one of U_i is nonzero, and let $\{\psi_j\}_{j=1}^l$ be an orthonormal basis of \mathcal{V} with $l = \dim \mathcal{V}$. Therefore, for each $U_i \in \mathcal{V}$ we will have

$$U_i = \sum_{j=1}^l (U_i, \psi_j)_{H_0^1(D)} \psi_j, \quad (12)$$

where $(U_i, \psi_j)_{H_0^1(D)} = (\nabla u_h^{m_i}, \nabla \psi_j)$.

Definition 4.1. *The POD method consists of finding an orthonormal basis ψ_j ($j = 1, 2, \dots, d$) such that for every $d = 1, \dots, l$, the following problem is solved*

$$\min_{\{\psi_j\}_{j=1}^d} \frac{1}{\ell} \sum_{i=1}^{\ell} \|U_i - \sum_{j=1}^d (U_i, \psi_j)_{H_0^1(D)} \psi_j\|_{H_0^1(D)}^2. \tag{13}$$

A solution $\{\psi_j\}_{j=1}^d$ of this minimization problem is known as a *POD basis of rank d* .

Let us introduce the correlation matrix $K = (K_{ij})_{i,j=1}^{\ell} \in \mathbb{R}^{\ell \times \ell}$ given by

$$K_{ij} = \frac{1}{\ell} (U_i, U_j)_{H_0^1(D)}. \tag{14}$$

The following proposition (see [46, 51, 52]) solves problem (13).

Proposition 4.1. *Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ denote the positive eigenvalues of K and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_l$ the associated orthonormal eigenvectors. Then a POD basis of rank $d \leq l$ is given by*

$$\psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{\ell} (\mathbf{v}_i)_j U_j, \quad i = 1, \dots, d, \tag{15}$$

where $(\mathbf{v}_i)_j$ denotes the j th component of the eigenvector \mathbf{v}_i . Furthermore, the following error formula holds:

$$\frac{1}{\ell} \sum_{i=1}^{\ell} \|U_i - \sum_{j=1}^d (U_i, \psi_j)_{H_0^1(D)} \psi_j\|_{H_0^1(D)}^2 = \sum_{j=d+1}^l \lambda_j. \tag{16}$$

Let $X^d := \text{span}\{\psi_1, \psi_2, \dots, \psi_d\}$, and consider the problem of finding $w_d^m \in X^d \subset H_h(D)$ such that $w_d^0(\mathbf{x}) = 0$ and for $m = 1, \dots, N$,

$$(w_d^m, v_d) + k(\nabla w_d^m, \nabla v_d) = k(g^m, v_d) + (w_d^{m-1}, v_d), \quad \forall v_d \in X^d \subset H_h(D), \quad m = 1, \dots, N. \tag{17}$$

Remark 4.1. *If \mathfrak{T}_h is a uniformly regular triangulation and $H_h(D)$ is the space of piecewise linear functions, the total degrees of freedom for problem (10) is N_h , where N_h is the number of vertices of triangles in \mathfrak{T}_h , while the total of degrees of freedom for problem (17) is d (where $d \ll l \ll \ell \ll N_h$).*

The following proposition, proved in [71], gives us an error estimate on the solution of problem (17).

Proposition 4.2. *If $w_h^m \in H_h(D)$ is the solution of problem (10), $w_d^m \in X^d \subset H_h(D)$ is the solution of problem (17), $k = O(h)$, $\ell^2 = O(N)$, and snapshots are equably taken, then for $m = 1, 2, \dots, N$, the following estimates hold:*

$$\|w_h^m - w_d^m\|_0 + \frac{1}{\ell} \sum_{j=1}^{\ell} \|\nabla(w_h^{m_j} - w_d^{m_j})\|_0 \leq C \left(k^{1/2} \sum_{j=d+1}^l \lambda_j \right)^{1/2}, \quad m = m_i, \quad i = 1, \dots, \ell; \tag{18}$$

$$\|w_h^m - w_d^m\|_0 + \frac{1}{\ell} \left[\|\nabla(w_h^m - w_d^m)\|_0 + \sum_{j=1}^{\ell-1} \|\nabla(w_h^{m_j} - w_d^{m_j})\|_0 \right] \leq C \left(k^{1/2} \sum_{j=d+1}^l \lambda_j \right)^{1/2} + Ck, \quad m \neq m_i.$$

Combining (11) and (18) we get the following result.

Proposition 4.3. *Under assumptions of Proposition 4.2, the error estimate between the solutions of problems (9) and (17), for $m = 1, 2, \dots, N$, is given by*

$$\|w^m - w_d^m\|_0 \leq Ch^{s+1} + Ck + C \left(k^{1/2} \sum_{j=d+1}^l \lambda_j \right)^{1/2} \tag{19}$$

Now, with a slight misuse of notation, we assume that the function f is given by $f = f(\mathbf{y})$, where $f \in C(\Gamma; C(0, T; L^2(D)))$ is the function employed in Eq. (5), and consider the following problem: find $u \in H_0^1(D)$ such that $u(\mathbf{x}, 0) = 0$, for all $\mathbf{x} \in D$, and

$$(u_t, v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(D). \quad (20)$$

Remark 4.2. Note that since $\|\mathbf{y} - \mathbf{y}'\| < \eta$ and under the assumption that $f \in C(\Gamma; C(0, T; L^2(D)))$ is Lipschitz continuous on Γ , we get that $\|f(\mathbf{y}) - f(\mathbf{y}')\|_{C(0, T; L^2(D))} = \|f - g\|_{C(0, T; L^2(D))} \leq L_f \|\mathbf{y} - \mathbf{y}'\|$, where L_f is the Lipschitz constant. Also, note that we are slightly misusing the symbol f to denote both the function $f \in C(\Gamma; C(0, T; L^2(D)))$ employed in Eq. (5) and the function $f = f(\mathbf{y}') \in C(0, T; L^2(D))$ used in Eq. (9).

Let us also consider the following problem: find $u_d^m \in X^d \subset H_h(D)$ such that $u_d^0(\mathbf{x}) = 0$ and for $m = 1, \dots, N$,

$$(u_d^m, v_d) + k(\nabla u_d^m, \nabla v_d) = k(f^m, v_d) + (u_d^{m-1}, v_d), \quad \forall v_d \in X^d \subset H_h(D), \quad m = 1, \dots, N. \quad (21)$$

Note that Eqs. (21) and (6) are identical, using the fact that we are using $f = f(\mathbf{y})$. Our aim is to find an estimate for $\|u^m - u_d^m\|_0$. First we need to prove two lemmas.

Lemma 4.1. Let u be the solution of problem (20) and let w be the solution of problem (9), then we have:

$$\|u^m - w^m\|_0 \leq C \|f - g\|_{C(0, T; L^2(D))}. \quad (22)$$

Proof. let $z = u - w$ and subtract Eqs. (9) and (20) to get:

$$(z_t, v) + (\nabla z, \nabla v) = (f - g, v), \quad \forall v \in H_0^1(D), \quad (23)$$

with $z(\mathbf{x}, 0) = 0$, for all $\mathbf{x} \in D$. Letting $v = z$ and integrating Eqs. (23) from 0 to t_m , we get

$$\frac{1}{2} \int_0^{t_m} \frac{d}{dt} \|z\|_0^2 dt + \int_0^{t_m} (\nabla z, \nabla z) dt = \int_0^{t_m} (f - g, z) dt.$$

This results in

$$\frac{1}{2} \|z^m\|_0^2 \leq \int_0^{t_m} \|f - g\|_0 \|z\|_0 dt \leq \frac{1}{2} \int_0^T \|f - g\|_0^2 dt + \frac{1}{2} \int_0^T \|z\|_0^2 dt.$$

Therefore,

$$\|z^m\|_0^2 \leq T \|f - g\|_{C(0, T; L^2(D))}^2 + \int_0^T \|z\|_0^2 dt. \quad (24)$$

Now we need to bound $\int_0^T \|z\|_0^2 dt$. For this, we integrate (23) once again but this time up to T , and use the Poincaré inequality $\|v\|_0 \leq C_p \|\nabla v\|_0$, for each $v \in H_0^1(D)$, to get

$$\frac{1}{2} \|z(T)\|_0^2 + \frac{1}{C_p^2} \int_0^T \|z\|_0^2 dt \leq \int_0^T \|f - g\|_0 \|z\|_0 dt.$$

Therefore,

$$\int_0^T \|z\|_0^2 dt \leq C_p^2 \left(\frac{1}{2\delta} \int_0^T \|f - g\|_0^2 dt + \frac{\delta}{2} \int_0^T \|z\|_0^2 dt \right).$$

Thus,

$$\left(1 - \frac{C_p^2}{2} \delta\right) \int_0^T \|z\|_0^2 dt \leq \frac{C_p^2}{2\delta} T \|f - g\|_{C(0, T; L^2(D))}^2.$$

Choose $\delta > 0$ such that $1 - (C_p^2/2)\delta > 0$, and let

$$C = \sqrt{T \left(1 + \frac{C_p^2}{2\delta - C_p^2 \delta^2}\right)}.$$

Now, Eq. (24) implies (22). □

Lemma 4.2. Let u_d^m be the solution of problem (21) and w_d^m be the solution of problem (17), then we have

$$\|u_d^m - w_d^m\|_0 \leq C \|f - g\|_{C(0,T;L^2(D))} \quad (25)$$

Proof. let $z_d^m = u_d^m - w_d^m$ and subtract Eqs. (17) and (21) to get:

$$(z_d^m, v_d) + k(\nabla z_d^m, \nabla v_d) = k(f^m - g^m, v_d) + (z_d^{m-1}, v_d), \quad \forall v_d \in X^d \subset H_h(D), \quad m = 1, \dots, N, \quad (26)$$

with $z_d^0(\mathbf{x}) = 0$. Let $v_d = z_d^m$ in Eq. (26) and use Poincaré inequality $\|v\|_0 \leq C_p \|\nabla v\|_0$, for each $v \in H_0^1(D)$, to achieve

$$\|z_d^m\|_0^2 + k \frac{1}{C_p^2} \|z_d^m\|_0^2 \leq k \|f^m - g^m\|_0 \|z_d^m\|_0 + \|z_d^{m-1}\|_0 \|z_d^m\|_0.$$

Therefore,

$$\left(1 + k \frac{1}{C_p^2}\right) \|z_d^m\|_0 \leq k \|f^m - g^m\|_0 + \|z_d^{m-1}\|_0,$$

which upon summation yields

$$\|z_d^m\|_0 \leq k \|f - g\|_{C(0,T;L^2(D))} \sum_{j=1}^m \left(\frac{1}{1 + (k/C_p^2)}\right)^j.$$

Let $\gamma = 1/C_p^2$ and note that $(1 + \gamma k)^m \leq e^{\gamma k m}$. Moreover, setting $\zeta = 1/(1 + \gamma k)$ we find

$$k \sum_{j=1}^m \left(\frac{1}{1 + (k/C_p^2)}\right)^j = k \frac{1 - \zeta^m}{\zeta^{-1} - 1} = \frac{1 - \zeta^m}{\gamma} \leq \frac{1 - e^{-\gamma k m}}{\gamma}.$$

Letting $C = (1 - e^{-\gamma k m})/\gamma$, we get (25). □

Now using estimates (19), (22), and (25) and Remark 4.2, we get the following error estimate.

Theorem 4.1. Let u be the solution of problem (20), and u_d^m be the solution of problem (21), for $m = 1, \dots, N$, we have

$$\|u^m - u_d^m\|_0 \leq C\eta + Ch^{s+1} + Ck + C \left(k^{1/2} \sum_{j=d+1}^l \lambda_j \right)^{1/2}, \quad (27)$$

where the eigenvalues λ_j depend on $\mathbf{y}' \in B_\eta(\mathbf{y}) \subset \Gamma$, and the constants C depend on \mathbf{y} and \mathbf{y}' , but are independent of h , k , and η .

5. ERROR ANALYSIS

In this section, we carry out an error analysis for the multifidelity collocation method introduced in Section 3 for problem (5). In [22], the authors showed that the collocation scheme (8) attains an exponential error decay for $u_d^m - u_{d,\mathbf{p}}^m$ with respect to each p_n , provided that the solution of (5) is analytic with respect to the random parameters. The convergence proof in [22] applies directly to our case. Therefore, in what follows, we prove the analyticity of the POD solution u_d^m with respect to each random variable y_n . This proof enables us to just state the corresponding convergence results.

5.1 Regularity Assumptions

Before going through the convergence analysis we need to impose some regularity assumptions on the forcing term f and the joint probability density function ρ , as in [22, 66]. In particular, we assume f to be continuous with respect to $\mathbf{y} \in \Gamma$ and that its growth at infinity is at most exponential, whenever the domain Γ is unbounded. In order to make it precise, we use the weight function $\boldsymbol{\sigma}(\mathbf{y}) = \prod_{n=1}^r \sigma_n(y_n) \leq 1$ introduced in [22], where $\sigma_n(y_n) = 1$ for each $y_n \in \Gamma_n$ whenever Γ_n is bounded. Moreover, if Γ_n is unbounded, we assume that $\sigma_n(y_n) = e^{-\alpha_n |y_n|}$ for some $\alpha_n > 0$. We also employ the space $C_{\boldsymbol{\sigma}}^0(\Gamma; V)$ of all continuous functions $v : \Gamma \rightarrow V$ such that

$$\max_{\mathbf{y} \in \Gamma} \{ \boldsymbol{\sigma}(\mathbf{y}) \|v(\mathbf{y})\|_V \} < +\infty,$$

where V is a Banach space. In what follows, we assume that $f \in C_{\boldsymbol{\sigma}}^0(\Gamma; C([0, T]; L^2(D)))$. We further assume that the joint density function ρ behaves like a Gaussian kernel at infinity. More precisely, we are assuming that there exist a constant $C_{\rho} > 0$ such that

$$\rho(\mathbf{y}) \leq C_{\rho} e^{-\sum_{n=1}^r (\delta_n y_n)^2}, \quad \forall \mathbf{y} \in \Gamma, \quad (28)$$

where δ_n is strictly positive if Γ_n is unbounded and zero otherwise. Under these assumptions, the following proposition is immediate; see [22].

Proposition 5.1. *The solution of problem (5) satisfies $u \in C_{\boldsymbol{\sigma}}^0(\Gamma; C(0, T; H_0^1(D)))$ and correspondingly, the approximate solution u_d^m resulted from (21) or equivalently (6), satisfies $u_d^m \in C_{\boldsymbol{\sigma}}^0(\Gamma; H_h(D))$, for $m = 1, \dots, N$.*

Furthermore, we have the following regularity result.

Lemma 5.1. *The following energy estimate holds:*

$$\|u_d^m\|_{L^2(D) \otimes L_p^2(\Gamma)} \leq C_p^2 \left(1 - e^{-(km/C_p^2)}\right) \|f\|_{C(0, T; L^2(D)) \otimes L_p^2(\Gamma)},$$

where C_p is the Poincaré constant.

Proof. Similar to the proof of Lemma 4.2. □

5.2 Regularity of the POD Solution

In this section we prove that whenever $f(\mathbf{y})$ is analytic and infinitely differentiable with respect to each component of \mathbf{y} , the solution u_d^m of Eq. (21) will be analytic with respect to each random parameter $y_n \in \Gamma$. To do this, we introduce the following notations as in [22, 66]:

$$\mathbf{y}_n^* \in \Gamma_n^* = \prod_{j=1, j \neq n}^r \Gamma_j,$$

and

$$\boldsymbol{\sigma}_n^* = \prod_{j=1, j \neq n}^r \sigma_j.$$

Lemma 5.2 (Lemma 3.2 of [22]). *Under the assumption that for every $\mathbf{y} = (y_n, \mathbf{y}_n^*) \in \Gamma$, there exists $\gamma_n < +\infty$ such that*

$$\frac{\|\partial_{y_n}^j f(\mathbf{y})\|_{C(0, T; L^2(D))}}{1 + \|f(\mathbf{y})\|_{C(0, T; L^2(D))}} \leq \gamma_n^j j!, \quad (29)$$

if the solution $u_d^m(\mathbf{x}, y_n, \mathbf{y}_n^*)$ is considered as a function of y_n , i.e., $u_d^m : \Gamma_n \rightarrow C_{\boldsymbol{\sigma}_n^*}^0(\Gamma_n^*; L^2(D))$, then the j th derivative of $u_d^m(\mathbf{x}, \mathbf{y})$ with respect to y_n satisfies

$$\|\partial_{y_n}^j u_d^m(\mathbf{y})\|_{L^2(D)} \leq C j! \gamma_n^j, \quad m = 1, \dots, N, \quad (30)$$

where C depends on $\|f(\mathbf{y})\|_{C(0, T; L^2(D))}$, and the Poincaré constant C_p .

Proof. Take the j th derivative of formulation (21) or equivalently (6) with respect to y_n , and let $v_d = \partial_{y_n}^j u_d^m(\mathbf{y})$ to get

$$\|\partial_{y_n}^j u_d^m(\mathbf{y})\|_0^2 + k\|\partial_{y_n}^j \nabla u_d^m(\mathbf{y})\|_0^2 = k(\partial_{y_n}^j f^m(\mathbf{y}), \partial_{y_n}^j u_d^m(\mathbf{y})) + (\partial_{y_n}^j u_d^{m-1}(\mathbf{y}), \partial_{y_n}^j u_d^m(\mathbf{y})).$$

Therefore,

$$\left(1 + \frac{k}{C_p^2}\right) \|\partial_{y_n}^j u_d^m(\mathbf{y})\|_0 \leq k\|\partial_{y_n}^j f^m(\mathbf{y})\|_0 + \|\partial_{y_n}^j u_d^{m-1}(\mathbf{y})\|_0,$$

which upon summation yields

$$\|\partial_{y_n}^j u_d^m(\mathbf{y})\|_0 \leq k\|\partial_{y_n}^j f(\mathbf{y})\|_{C(0,T;L^2(D))} \sum_{i=1}^m \left(\frac{1}{1 + (k/C_p^2)}\right)^i.$$

Thus,

$$\|\partial_{y_n}^j u_d^m(\mathbf{y})\|_0 \leq C_p^2 \left(1 - e^{-(km/C_p^2)}\right) [1 + \|f(\mathbf{y})\|_{C(0,T;L^2(D))}] \gamma_n^j j!.$$

Letting $C = C_p^2 \left(1 - e^{-(km/C_p^2)}\right) [1 + \|f(\mathbf{y})\|_{C(0,T;L^2(D))}]$ we get (30). \square

We will immediately obtain the following theorem, whose proof closely follows the proof of Theorem 4.4 in [66].

Theorem 5.1 (Theorem 4.4 in [66]). *Under assumption (29), the solution $u_d^m(\mathbf{x}, y_n, \mathbf{y}_n^*)$ considered as a function of y_n , admits an analytic extension $u_d^m(\mathbf{x}, z, \mathbf{y}_n^*)$, $z \in \mathbb{C}$, in the region of complex plane*

$$\Sigma(\Gamma_n, \tau_n) := \{z \in \mathbb{C} : \text{dist}(z, \Gamma_n) \leq \tau_n\},$$

where $0 < \tau_n < 1/\gamma_n$.

5.3 Convergence Analysis

It can be noted that the total error $e^m = u^m - u_{d,\mathbf{p}}^m$ can be written as $e^m = (u^m - u_d^m) + (u_d^m - u_{d,\mathbf{p}}^m)$. Therefore, one can express *a priori* estimates for the error e^m in the $L^2(D) \otimes L_p^2(\Gamma)$ norm in terms of the corresponding error estimates for $(u^m - u_d^m)$ and $(u_d^m - u_{d,\mathbf{p}}^m)$. One can estimate the interpolation error $(u_d^m - u_{d,\mathbf{p}}^m)$ by repeating the same procedure as in [22], using the analyticity result of Theorem 5.1. Please refer to Section 4 of [22] for more details.

Theorem 5.2 (Theorem 4.1 in [22]). *Under assumption (29), there exist positive constants b_n , $n = 1, \dots, r$, and C that are independent of h , d , and \mathbf{p} such that*

$$\|u_d^m - u_{d,\mathbf{p}}^m\|_{L^2(D) \otimes L_p^2(\Gamma)} \leq C \sum_{n=1}^r \beta_n(p_n) \exp(-b_n p_n^{\theta_n}), \quad (31)$$

where

$$\begin{cases} \theta_n = \beta_n = 1, \\ b_n = \log \left[\frac{2\tau_n}{|\Gamma_n|} \left(1 + \sqrt{1 + \frac{|\Gamma_n|^2}{4\tau_n^2}} \right) \right], \end{cases}$$

if Γ_n is bounded, and

$$\begin{cases} \theta_n = \frac{1}{2}, & \beta_n = O(\sqrt{p_n}), \\ b_n = \tau_n \delta_n, \end{cases}$$

if Γ_n is unbounded, with τ_n being the minimum distance between Γ_n and the nearest singularity in the complex plane, as defined in theorem 5.1, and δ_n is defined in assumption (28).

Remark 5.1. For an isotropic full tensor-product approximation, i.e., $p_1 = p_2 = \dots = p_r = p$, the number of collocation points Θ is given by $\Theta = (1 + p)^r$. Thus, one can easily obtain the following error bound with respect to Θ ; see [66]:

$$\|u_d^m - u_{d,\mathbf{p}}^m\|_{L^2(D) \otimes L_\rho^2(\Gamma)} \leq \begin{cases} C\Theta^{-b_{\min}/r}, & \text{if } \Gamma \text{ is bounded,} \\ C\Theta^{-b_{\min}/2r}, & \text{if } \Gamma \text{ is unbounded,} \end{cases} \quad (32)$$

where $b_{\min} = \min\{b_1, b_2, \dots, b_r\}$ as in Theorem 5.2. The constant C does not depend on r and b_{\min} . Note that for large values of r , sparse grid stochastic collocation methods [26, 72], specially adaptive and anisotropic ones, e.g., [25, 27] are more effective in dealing with the curse of dimensionality evident in the inequality (32). This inequality shows that as the dimension r increases, the convergence becomes slower. The analyticity result (Theorem 5.1) combined with the analysis in [25–27, 72], can easily lead to the derivation of error bounds for sparse grid approximations. For instance, the error in an isotropic Smolyak approximation [26, 72] with a total of Θ sparse grid points, can be bounded by

$$C\Theta^{-b_{\min}/(1+\log(2r))}.$$

Here, we will give a short description of the isotropic Smolyak algorithm. More detailed information can be found in [26, 73]. Assume $p_1 = p_2 = \dots = p_r = p$. For $r = 1$, let $\{\mathcal{I}_{1,i}\}_{i=1,2,\dots}$ be a sequence of interpolation operators given by Eq. (8). Define $\Delta_0 = \mathcal{I}_{1,0} = 0$ and $\Delta_i = \mathcal{I}_{1,i} - \mathcal{I}_{1,i-1}$. Now for $r > 1$, let

$$\mathcal{A}(q, r) = \sum_{0 \leq i_1 + i_2 + \dots + i_r \leq q} \Delta_{i_1} \otimes \dots \otimes \Delta_{i_r}, \quad (33)$$

where q is a non-negative integer. $\mathcal{A}(q, r)$ is the Smolyak operator, and q is known as the sparse grid level.

Now we need to find error bounds for the deterministic part of our algorithm in the $L^2(D) \otimes L_\rho^2(\Gamma)$ norm, i.e., $u^m - u_d^m$. First, note that according to (28), the joint density function ρ behaves like a Gaussian kernel at infinity. Therefore, in practice we are literally dealing with a compact random parameter set Γ , since we can approximate Γ with a large enough compact set. So from now on we assume that Γ is compact. We know that $\Gamma \subset \bigcup_{\mathbf{y}' \in \Gamma} B_\eta(\mathbf{y}')$. Thus, using the compactness assumption on Γ , there exist $\Upsilon \in \mathbb{N}_+$ and $\{\mathbf{y}'^i\}_{i=1}^\Upsilon \subset \Gamma$ such that $\Gamma = \bigcup_{i=1}^\Upsilon B_\eta(\mathbf{y}'^i) \cap \Gamma$. Letting ${}^i\Gamma = B_\eta(\mathbf{y}'^i) \cap \Gamma$, we can write $\Gamma = \bigcup_{i=1}^\Upsilon {}^i\Gamma$.

Theorem 5.3. Under the Lipschitz continuity (see Remark 4.2) assumption, there exist constants C and Λ such that

$$\|u^m - u_d^m\|_{L^2(D) \otimes L_\rho^2(\Gamma)} \leq C\eta + Ch^{s+1} + Ck + Ck^{1/4}\Lambda. \quad (34)$$

Proof. Let us first integrate the the last term in estimate (27). Thus, we have

$$\begin{aligned} & \int_\Gamma \left\{ C(\mathbf{y}, \mathbf{y}'(\mathbf{y})) \left(k^{1/2} \sum_{j=d(\mathbf{y}, \mathbf{y}'(\mathbf{y}))+1}^{l(\mathbf{y}, \mathbf{y}'(\mathbf{y}))} \lambda_j(\mathbf{y}'(\mathbf{y})) \right)^{1/2} \right\}^2 \rho(\mathbf{y}) d\mathbf{y} \\ &= k^{1/2} \int_\Gamma C(\mathbf{y}, \mathbf{y}'(\mathbf{y}))^2 \sum_{j=d(\mathbf{y}, \mathbf{y}'(\mathbf{y}))+1}^{l(\mathbf{y}, \mathbf{y}'(\mathbf{y}))} \lambda_j(\mathbf{y}'(\mathbf{y})) \rho(\mathbf{y}) d\mathbf{y} \\ &= k^{1/2} \sum_{i=1}^\Upsilon \left(\sum_{j=d({}^i\mathbf{y}')+1}^{l({}^i\mathbf{y}')} \lambda_j({}^i\mathbf{y}') \right) \int_{{}^i\Gamma} C(\mathbf{y}, {}^i\mathbf{y}')^2 \rho(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

Now letting $\Lambda_i = \sum_{j=d({}^i\mathbf{y}')+1}^{l({}^i\mathbf{y}')} \lambda_j({}^i\mathbf{y}')$, and assuming $\Lambda^2 = \max_{i=1, \dots, \Upsilon} \{\Lambda_i\}$, we get the following upper bound for the above expression:

$$k^{1/2} \Lambda^2 \sum_{i=1}^\Upsilon \int_{{}^i\Gamma} C(\mathbf{y}, {}^i\mathbf{y}')^2 \rho(\mathbf{y}) d\mathbf{y} = k^{1/2} \Lambda^2 \int_\Gamma C(\mathbf{y}, \mathbf{y}'(\mathbf{y}))^2 \rho(\mathbf{y}) d\mathbf{y}.$$

Letting $C^2 = \int_{\Gamma} C(\mathbf{y}, \mathbf{y}')^2 \rho(\mathbf{y}) d\mathbf{y}$, we get the last term in (34). The first three terms of (34) can also be easily computed by integrating the first three terms of (27). We will get the same expressions for the constants C as above. \square

Remark 5.2. Note that due to the way that the POD method works, the constant Λ is so small that the $k^{1/4}$ term has a very little effect on the error.

Combining (31) and (34), we will finally get the following total error estimate.

Theorem 5.4. Under assumption (29) and the Lipschitz continuity (see Remark 4.2) assumption, there exist positive constants C and Λ that are independent of h, k, η , and \mathbf{p} , and there exist constants $b_n, n = 1, \dots, r$, such that

$$\|u^m - u_{d,\mathbf{p}}^m\|_{L^2(D) \otimes L_p^2(\Gamma)} \leq C\eta + Ch^{s+1} + Ck + Ck^{1/4}\Lambda + C \sum_{n=1}^r \beta_n(p_n) \exp(-b_n p_n^{\theta_n}), \quad (35)$$

where θ_n, β_n , and p_n are the same as the ones in Theorem 5.2.

Remark 5.3. In some cases, one might be interested in estimating the expectation error; i.e., $\|\mathbb{E}[u^m - u_{d,\mathbf{p}}^m]\|_{L^2(D)}$. This can be easily achieved by observing that

$$\|\mathbb{E}[u^m - u_{d,\mathbf{p}}^m]\|_{L^2(D)}^2 \leq \|u^m - u_{d,\mathbf{p}}^m\|_{L^2(D) \otimes L_p^2(\Gamma)}^2. \quad (36)$$

6. NUMERICAL EXPERIMENTS

In this section, we provide a computational example to illustrate the advantages of multifidelity stochastic collocation method. Specifically, we consider problem (1) with $D = (0, 1)^2 \subset \mathbb{R}^2$, $T = 1$, and the forcing term being given by

$$f(t, \mathbf{x}, \omega) = 10 + e^t \sum_{n=1}^r \mathbf{y}_n(\omega) \sin(n\pi x).$$

The real-valued random variables $y_n, n = 1, \dots, r$, are supposed to be independent and have uniform distributions $U(0, 1)$. In the following, we let $r = 4$. We employ the sparse grid stochastic collocation method introduced in Remark 5.1 with sparse grid level $q = 8$. We use the Clenshaw-Curtis abscissas (see [74]) as collocation points. These abscissas are the extrema of Chebyshev polynomials. We divide the spatial domain D into 32×32 small squares with side length $\Delta x = \Delta y = 1/32$, and then we connect the diagonals of the squares to divide each square into two triangles. These triangles consist the triangulation \mathfrak{T}_h , with $h = \sqrt{2}/32$. Take $k = 0.1$ as the time step increment. We use all of the time steps to form the snapshots. We employ six POD basis functions. In the following, we compare the solution resulting from a regular isotropic sparse grid stochastic collocation method which only uses the finite element method, with the hybrid multifidelity method proposed in this paper which employs both finite element and POD methods. In Fig. 1, we compare the expected values resulting from the multifidelity method and a regular sparse grid stochastic collocation method. We take $\eta = 0.1$. Recall that for each $\mathbf{y} \in \Gamma$ our method searches the η neighborhood of \mathbf{y} to check whether for some $\mathbf{y}' \in B_\eta(\mathbf{y})$ problem (5) is already solved. If a nearby problem (at \mathbf{y}') is found to be solved by the finite element method, our algorithm uses this information to create POD basis functions and solves problem (5) at \mathbf{y} using Galerkin-POD method, which is computationally much cheaper than finite element. Moreover, Fig. 2 compares variances of solutions resulting from the two methods.

Figures 3 and 4, show the convergence patterns of expectations and variances of solutions with regard to η , respectively. These results validate our theoretical estimates of previous sections in the sense that they justify the presence of the $C\eta$ term in Theorem 5.4. We are actually comparing our multifidelity method with a regular sparse grid stochastic method. Note that for small enough η (less than the shortest distance between the collocation points) we get the regular sparse grid method back. Therefore the error is zero for such a small η .

Figure 5 demonstrates how the number of times that the finite element code is employed increases with respect to a decrease in η .

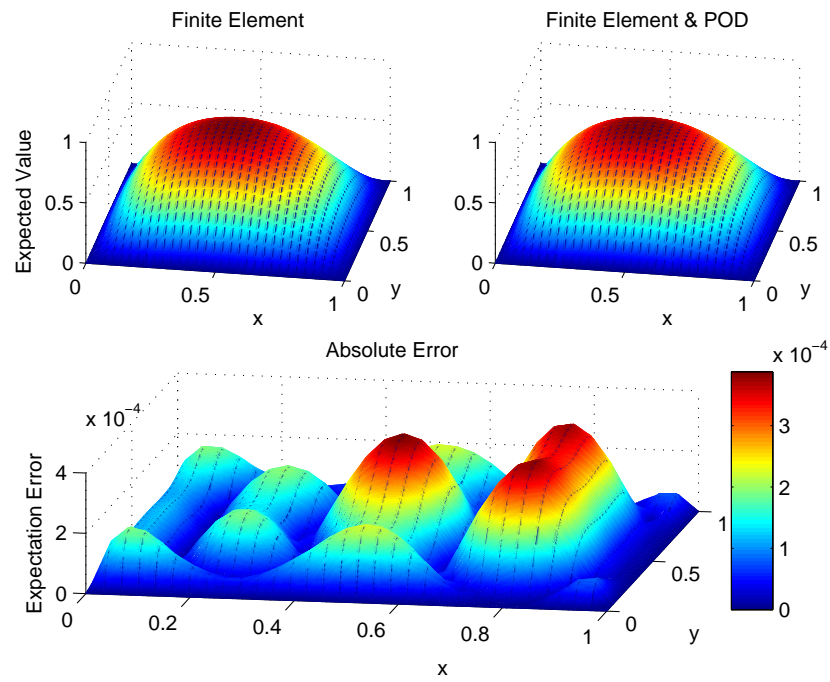


FIG. 1: Comparison of expected values (bottom) resulting from a regular sparse grid method (top left) and the multi-fidelity method with $\eta = 0.1$ (top right).

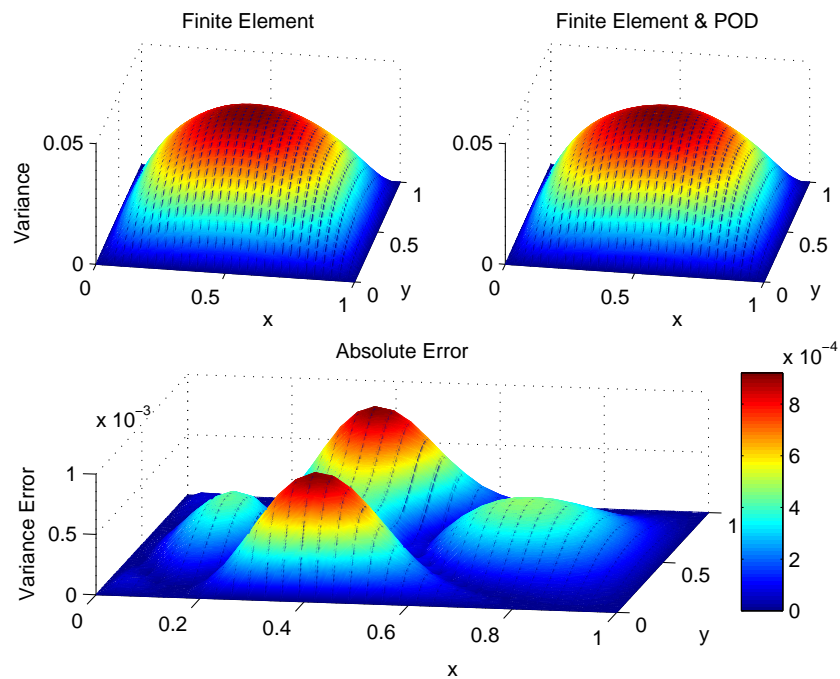


FIG. 2: Comparison of variances of solutions (bottom) resulting from a regular sparse grid method (top left) and the multi-fidelity method with $\eta = 0.1$ (top right).

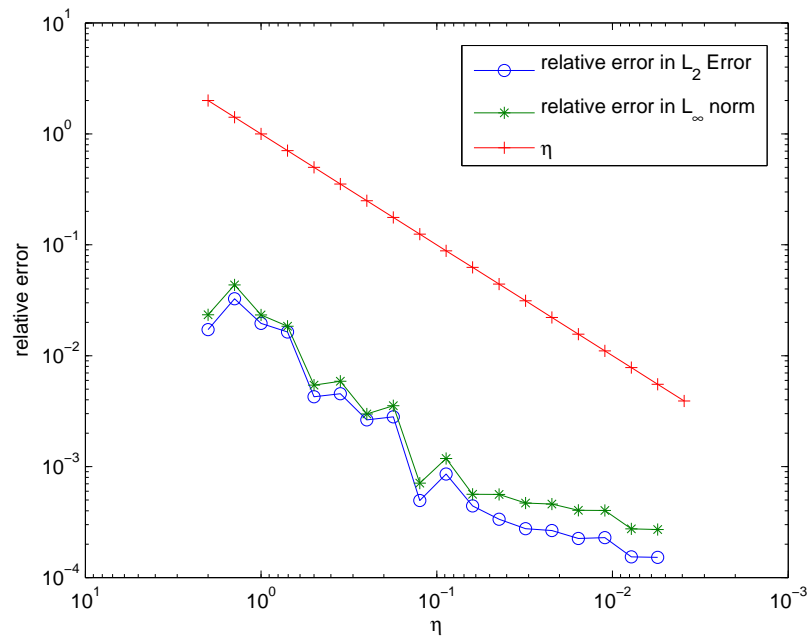


FIG. 3: Convergence pattern of expected values of solutions with respect to η .

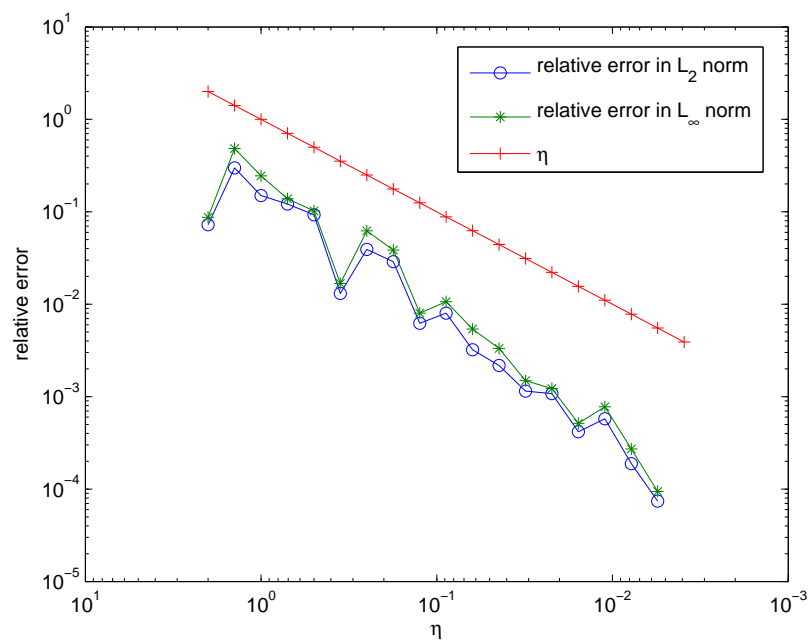


FIG. 4: Convergence pattern of variances of solutions with respect to η .

Table 1 summarizes the results when $\eta = 0.1$. In this case, the number of times that the finite element code is utilized by the multifidelity method is 3745. Compared it to 18,946, the number of times that a regular sparse grid calls the finite element code.

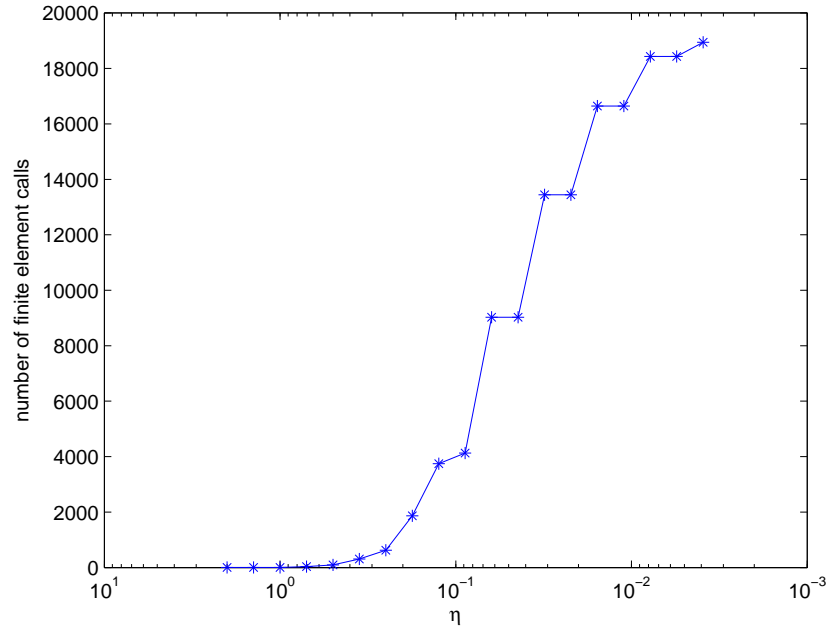


FIG. 5: The number of times that the finite element code is employed as a function of η .

TABLE 1: Relative errors when $\eta = 0.1$

	Relative error in L_2 norm	Relative error in L_∞ norm
Expected value	3.6×10^{-4}	4.8×10^{-4}
Variance	1.2×10^{-2}	2.0×10^{-2}

Table 2 is just another way of presenting the data depicted in Figs. 3–5.

Remark 6.1. *This method with some slight improvements using sensitivity analysis of POD basis functions is applied to the stochastic Burgers equation driven by Brownian motion in [75]. Similar performances are observed in that paper.*

7. CONCLUDING REMARKS

In this paper, we have proposed a method to enhance the performance of stochastic collocation methods using proper orthogonal decomposition. We have carried out detailed error analyses of the proposed multifidelity stochastic collocation methods for parabolic partial differential equations with random forcing terms. We illustrated and supported our theoretical analyses with a numerical example. The analysis of this paper can be simply generalized to parabolic partial differential equations with random initial conditions and random coefficients. Our method only requires a well-posedness argument of the corresponding deterministic equations. Future works in this area can include applications of this method to partial differential equations in fluid mechanics, and proving error estimates for these equations.

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TABLE 2: Relative errors and the number of times that the finite element code is employed for different values of η

η	# FE calls	Expectation L_2 error	Expectation L_∞ error	Variance L_2 error	Variance L_∞ error
4	1	1.72E-02	2.34E-02	7.25E-02	8.72E-02
2	3	3.27E-02	4.35E-02	2.99E-01	4.84E-01
1	5	1.95E-02	2.33E-02	1.50E-01	2.45E-01
1/2	36	1.63E-02	1.85E-02	1.21E-01	1.38E-01
(1/2) ²	92	4.26E-03	5.43E-03	9.27E-02	1.02E-01
(1/2) ³	306	4.55E-03	5.89E-03	1.31E-02	1.68E-02
(1/2) ⁴	621	2.64E-03	2.98E-03	3.91E-02	6.21E-02
(1/2) ⁵	1866	2.81E-03	3.55E-03	2.88E-02	3.86E-02
(1/2) ⁶	3743	4.96E-04	7.09E-04	6.23E-03	8.00E-03
(1/2) ⁷	4129	8.58E-04	1.19E-03	8.00E-03	1.07E-02
(1/2) ⁸	9026	4.42E-04	5.63E-04	3.22E-03	5.39E-03
(1/2) ⁹	9026	3.35E-04	5.61E-04	2.18E-03	3.33E-03
(1/2) ¹⁰	13442	2.76E-04	4.69E-04	1.15E-03	1.49E-03
(1/2) ¹¹	13442	2.65E-04	4.59E-04	1.08E-03	1.22E-03
(1/2) ¹²	16642	2.25E-04	4.04E-04	4.18E-04	5.15E-04
(1/2) ¹³	16642	2.29E-04	4.02E-04	5.75E-04	7.78E-04
(1/2) ¹⁴	18434	1.54E-04	2.75E-04	1.89E-04	2.71E-04
(1/2) ¹⁵	18434	1.52E-04	2.71E-04	7.42E-05	9.43E-05
(1/2) ¹⁶	18946	0	0	0	0

REFERENCES

1. Papanicolaou, G., Wave propagation in a one-dimensional random medium, *SIAM J. Appl. Math.*, 21(1):13–18, 1971.
2. Papanicolaou, G., Diffusion in random media, *Surveys Appl. Math.*, 1:205–253, 1995.
3. Bensoussan, A. and Temam, R., Equations stochastiques du type Navier-Stokes, *J. Funct. Anal.*, 13(2):195–222, 1973.
4. Da Prato, G. and Debussche, A., Ergodicity for the 3D stochastic Navier–Stokes equations, *J. Math. Pures Appl.*, 82(8):877–947, 2003.
5. Weinan, E., Khanin, K., Mazel, A., and Sinai, Y., Probability distribution functions for the random forced burgers equation, *Phys. Rev. Lett.*, 78:1904–1907, 1997.
6. Mikulevicius, R. and Rozovskii, B., Stochastic Navier–Stokes equations for turbulent flows, *SIAM J. Math. Anal.*, 35(5):1250–1310, 2004.
7. Cameron, R. and Martin, W., The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals, *An. Math.*, 48(2):385–392, 1947.
8. Hida, T., Kuo, H., Potthoff, J., and Streit, L., *White Noise: An Infinite Dimensional Calculus*, Vol. 253, Springer, Berlin, 1993.
9. Wiener, N., The homogeneous chaos, *Am. J. Math.*, 60(4):897–936, 1938.
10. Crow, S. and Canavan, G., Relationship between a wiener-hermite expansion and an energy cascade, *J. Fluid Mech.*, 41(2):387–403, 1970.
11. Orszag, S. and Bissonnette, L., Dynamical properties of truncated wiener-hermite expansions, *Phys. Fluids*, 10:2603–2613, 1967.
12. Chorin, A., Hermite expansions in monte-carlo computation, *J. Comput. Phys.*, 8(3):472–482, 1971.
13. Chorin, A., Gaussian fields and random flow, *J. Fluid Mech.*, 63:21–32, 1974.
14. Sakamoto, S. and Ghanem, R., Simulation of multi-dimensional non-gaussian non-stationary random fields, *Probabilistic Eng. Mech.*, 17(2):167–176, 2002.

15. Ghanem, R. and Spanos, P., *Stochastic Finite Elements: A Spectral Approach*, Dover Publications, New York, 2003.
16. Xiu, D. and Karniadakis, G., Modeling uncertainty in flow simulations via generalized polynomial chaos, *J. Comput. Phys.*, 187(1):137–167, 2003.
17. Zhang, D. and Lu, Z., An efficient, high-order perturbation approach for flow in random porous media via karhunen–loeve and polynomial expansions, *J. Comput. Phys.*, 194(2):773–794, 2004.
18. Jardak, M., Su, C., and Karniadakis, G., Spectral polynomial chaos solutions of the stochastic advection equation, *J. Sci. Comput.*, 17(1):319–338, 2002.
19. Xiu, D. and Karniadakis, G., The Wiener–Askey polynomial chaos for stochastic differential equations, *SIAM J. Sci. Comput.*, 24(2):619–644, 2002.
20. Mathelin, L. and Hussaini, M. Y., A stochastic collocation algorithm for uncertainty analysis, NASA, pp. 2003–212153, 2003.
21. Tatang, M., Pan, W., Prinn, R., and McRae, G., An efficient method for parametric uncertainty analysis of numerical geophysical models, *J. Geophys. Res.*, 102(D18):21925–21, 1997.
22. Babuška, I., Nobile, F., and Tempone, R., A stochastic collocation method for elliptic partial differential equations with random input data, *SIAM J. Num. Anal.*, 45(3):1005–1034, 2007.
23. Xiu, D. and Hesthaven, J., High-order collocation methods for differential equations with random inputs, *SIAM J. Sci. Comput.*, 27(3):1118–1139, 2005.
24. Agarwal, N. and Aluru, N., A domain adaptive stochastic collocation approach for analysis of mems under uncertainties, *J. Comput. Phys.*, 228(20):7662–7688, 2009.
25. Ma, X. and Zabaras, N., An adaptive hierarchical sparse grid collocation algorithm for the solution of stochastic differential equations, *J. Comput. Phys.*, 228(8):3084–3113, 2009.
26. Nobile, F., Tempone, R., and Webster, C. G., A sparse grid stochastic collocation method for partial differential equations with random input data, *SIAM J. Num. Anal.*, 46(5):2309–2345, 2008.
27. Nobile, F., Tempone, R., and Webster, C. G., An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data, *SIAM J. Num. Anal.*, 46(5):2411–2442, 2008.
28. Hou, T., Luo, W., Rozovskii, B., and Zhou, H., Wiener chaos expansions and numerical solutions of randomly forced equations of fluid mechanics, *J. Comput. Phys.*, 216(2):687–706, 2006.
29. Xiu, D. and Karniadakis, G., Supersensitivity due to uncertain boundary conditions, *Int. J. Num. Methods Eng.*, 61(12):2114–2138, 2004.
30. Le Maître, O. P., Knio, O., Najm H. N., and Ghanem R. G., Uncertainty propagation in CFD using polynomial chaos decomposition, *Fluid Dyn. Res.*, 38(9):616–640, 2006.
31. Knio, O., Najm, H., Ghanem, R., A stochastic projection method for fluid flow: I. Basic formulation, *J. Comput. Phys.*, 173(2):481–511, 2001.
32. Le Maître, O., Reagan, M., Najm, H., Ghanem, R., and Knio, O., A stochastic projection method for fluid flow: II. Random process, *J. Comput. Phys.*, 181(1):9–44, 2002.
33. Lin, G., Wan, X., Su, C., and Karniadakis, G., Stochastic computational fluid mechanics, *Comput. Sci. Eng.*, 9(2):21–29, 2007.
34. Xiu, D., Lucor, D., Su, C., and Karniadakis, G., Stochastic modeling of flow-structure interactions using generalized polynomial chaos, *J. Fluids Eng.*, 124:51–59, 2002.
35. Chen, Q., Gottlieb, D., and Hesthaven, J., Uncertainty analysis for the steady-state flows in a dual throat nozzle, *J. Comput. Phys.*, 204(1):378–398, 2005.
36. Gottlieb, D. and Xiu, D., Galerkin method for wave equations with uncertain coefficients, *Commun. Comput. Phys.*, 3(2):505–518, 2008.
37. Lin, G., Su, C., and Karniadakis, G., Predicting shock dynamics in the presence of uncertainties, *J. Comput. Phys.*, 217(1):260–276, 2006.
38. Doostan, A., Ghanem, R., and Red-Horse, J., Stochastic model reduction for chaos representations, *Comput. Methods Appl. Mech. Eng.*, 196(37):3951–3966, 2007.
39. Ghanem, R., Masri, S., Pellissetti, M., and Wolfe, R., Identification and prediction of stochastic dynamical systems in a polynomial chaos basis, *Comput. Methods Appl. Mech. Eng.*, 194(12):1641–1654, 2005.

40. Ghanem, R. and Doostan, A., On the construction and analysis of stochastic models: Characterization and propagation of the errors associated with limited data, *J. Comput. Phys.*, 217(1):63–81, 2006.
41. Canuto, C. and Kozubek, T., A fictitious domain approach to the numerical solution of PDEs in stochastic domains, *Num. Math.*, 107(2):257–293, 2007.
42. Lin, G., Su, C., and Karniadakis, G., Random roughness enhances lift in supersonic flow, *Phys. Rev. Lett.*, 99(10):104501, 2007.
43. Tartakovsky, D. and Xiu, D., Stochastic analysis of transport in tubes with rough walls, *J. Comput. Phys.*, 217(1):248–259, 2006.
44. Xiu, D. and Tartakovsky, D., Numerical methods for differential equations in random domains, *SIAM J. Sci. Comput.*, 28(3):1167–1185, 2006.
45. Robinson, T., Eldred, M., Willcox, K., and Haimes, R., Surrogate-based optimization using multifidelity models with variable parameterization and corrected space mapping, *AIAA J.*, 46(11):2814–2822, 2008.
46. Sirovich, L., Turbulence and the dynamics of coherent structures. I-Coherent structures. II-Symmetries and transformations. III-Dynamics and scaling, *Q. Appl. Math.*, 45:561–571, 1987.
47. Chambers, D., Adrian, R., Moin, P., Stewart, D., and Sung, H., Karhunen–Loève expansion of Burgers model of turbulence, *Phys. Fluids*, 31(9):2573–2582, 1988.
48. Holmes, P., Lumley, J., and Berkooz, G., *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge University Press, Cambridge, 1998.
49. Fahl, M., Computation of pod basis functions for fluid flows with lanczos methods, *Math. Comput. Model.*, 34(1):91–107, 2001.
50. Iollo, A., Lanteri, S., and Désidéri, J., Stability properties of POD–Galerkin approximations for the compressible Navier–Stokes equations, *Theor. Comput. Fluid Dyn.*, 13(6):377–396, 2000.
51. Kunisch, K. and Volkwein, S., Galerkin proper orthogonal decomposition methods for parabolic problems, *Numer. Math.*, 90(1):117–148, 2001.
52. Kunisch, K. and Volkwein, S., Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics, *SIAM J. Numer. Anal.*, 40(2):492–515, 2002.
53. Henri, T. and Yvon, J., *Stability of the POD and Convergence of the POD Galerkin Method for Parabolic Problems*, Université de Rennes, 2002.
54. Rowley, C., Colonius, T., and Murray, R., Model reduction for compressible flows using POD and Galerkin projection, *Phys. D: Nonlinear Phenom.*, 189(1):115–129, 2004.
55. Camphouse, R., Boundary feedback control using proper orthogonal decomposition models, *J. Guidance, Control, Dyn.*, 28(5):931–938, 2005.
56. Pearson, K., LIII. On lines and planes of closest fit to systems of points in space, *London, Edinburgh, Dublin Philos. Mag. J. Sci.*, 2(11):559–572, 1901.
57. Hotelling, H., Analysis of a complex of statistical variables into principal components., *J. Educ. Psychol.*, 24(6):417–441, 1933.
58. Björnsson, H. and Venegas, S., A manual for EOF and SVD analyses of climatic data, *CCGCR Rep.*, 97(1):52, 1997.
59. Ravindran, S., A reduced-order approach for optimal control of fluids using proper orthogonal decomposition, *Int. J. Numer. Methods Fluids*, 34(5):425–448, 2000.
60. Atwell, J., Borggaard, J., and King, B., Reduced order controllers for Burgers’ equation with a nonlinear observer, *Appl. Math. Comput. Sci.*, 11(6):1311–1330, 2001.
61. Atwell, J. and King, B., Proper orthogonal decomposition for reduced basis feedback controllers for parabolic equations, *Math. Comput. Model.*, 33(1):1–19, 2001.
62. Kepler, G., Banks, H., Tran, H., and Beeler, S., Reduced order modeling and control of thin film growth in an HPCVD reactor, *SIAM J. Appl. Math.*, 62(4):1251–1280, 2002.
63. Atwell, J. and King, B., Reduced order controllers for spatially distributed systems via proper orthogonal decomposition, *SIAM J. Sci. Comput.*, 26(1):128–151, 2004.

64. Lee, C. and Tran, H., Reduced-order-based feedback control of the Kuramoto–Sivashinsky equation, *J. Comput. Appl. Math.*, 173(1):1–19, 2005.
65. Henri, T. and Yvon, J., Convergence estimates of POD-Galerkin methods for parabolic problems, *Syst. Model. Optimization*, 166:295–306, 2005.
66. Zhang, G. and Gunzburger, M., Error analysis of a stochastic collocation method for parabolic partial differential equations with random input data, *SIAM J. Numer. Anal.*, 50(4):1922–1940, 2012.
67. Loeve, M., *Probability Theory, Volume I, Graduate Texts in Mathematics*, 4th edition, Springer-Verlag, New York, 1977.
68. Evans, L., *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 2002.
69. Thomée, V., *Galerkin Finite Element Methods for Parabolic Problems*, Vol. 25, Springer Verlag, Berlin, 1997.
70. Ciarlet, P. G., *The Finite Element Method for Elliptic Problems*, Vol. 4, North Holland, Amsterdam, 1978.
71. Luo, Z., Chen, J., Sun, P., and Yang, X., Finite element formulation based on proper orthogonal decomposition for parabolic equations, *Sci. China Ser. Math.*, 52(3):585–596, 2009.
72. Nobile, F. and Tempone, R., Analysis and implementation issues for the numerical approximation of parabolic equations with random coefficients, *Int. J. Numer. Methods Eng.*, 80(6-7):979–1006, 2009.
73. Barthelmann, V., Novak, E., and Ritter, K., High dimensional polynomial interpolation on sparse grids, *Adv. Comput. Math.*, 12(4):273–288, 2000.
74. Clenshaw, C. W. and Curtis, A. R., A method for numerical integration on an automatic computer, *Numer. Math.*, 2(1):197–205, 1960.
75. Raissi, M. and Seshaiyer, P., A multi-fidelity stochastic collocation method using locally improved reduced-order models, *Num. Anal.*, arXiv:1306.0132, 2013.