

CONVOLVED ORTHOGONAL EXPANSIONS FOR UNCERTAINTY PROPAGATION: APPLICATION TO RANDOM VIBRATION PROBLEMS

X. Frank Xu^{1,*} & George Stefanou²

¹Department of Civil, Environmental and Ocean Engineering, Stevens Institute of Technology, Hoboken, New Jersey 07030, USA

²Institute of Structural Analysis and Antiseismic Research, National Technical University of Athens, 15780 Athens, Greece

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Physical nonlinear systems are typically characterized with n -fold convolution of the Green's function, e.g., nonlinear oscillators, inhomogeneous media, and scattering theory in continuum and quantum mechanics. A novel stochastic computation method based on orthogonal expansions of random fields has been recently proposed [1]. In this study, the idea of orthogonal expansion is formalized as the so-called n th-order convolved orthogonal expansion (COE) method, especially in dealing with random processes in time. Although the paper is focused on presentation of the properties of the convolved random basis processes, examples are also provided to demonstrate application of the COE method to random vibration problems. In addition, the relation to the classical Volterra-type expansions is discussed

KEY WORDS: orthogonal expansion, non-Gaussian, random process, nonlinear dynamics

1. INTRODUCTION

The modeling of complex systems involving multiscale behavior, nonlinearity, and uncertainty is recognized as one of the most challenging engineering and scientific problems [2]. Uncertainty propagation in a complex system can be generally considered as a mapping of input–output probabilistic information through a system typically described as a differential equation. In solving stochastic differential equations, a random-variable-based polynomial chaos (rv-PC) method has been introduced as a major numerical solver [3–5]. However, for most realistic random process problems, the rv-PC method confronts a critical challenge of the curse of dimensionality, e.g., Refs. [6, 7]. Various efforts have been made in order to alleviate the severity of the curse of dimensionality, such as the stochastic collocation method using a biorthogonal basis [8] and its sparse grid version combined with the homogenization method [9, 10]. By applying a separated representation strategy onto the rv-PC method, recent works [11, 12] show that the combined approach can be competitive against Monte Carlo methods for certain examples with hundreds of dimensions. It is worth noting that a random-variable representation of weakly correlated random fields or processes is one of the major sources of the curse of dimensionality. Distinguished from all the random-variable-based methods, a novel random-field-based orthogonal expansion method has been recently proposed in [1] to circumvent the curse of dimensionality for many physical systems where the input information is represented as random fields or processes.

Physical nonlinear systems are typically characterized with n -fold convolution of Green's function, e.g., nonlinear oscillators (see Section 4), inhomogeneous media [13], and scattering theory in continuum [14] and quantum mechanics. In this study, the idea presented in [1] is further formalized as the so-called n th-order convolved orthogonal expansion (COE) method, especially in dealing with random processes in the time domain. Although the paper is focused on presentation of the properties of the convolved random basis processes, examples are also provided to

*Correspond to X. Frank Xu, E-mail: xxu1@stevens.edu

NOMENCLATURE			
ϕ_i	ith random basis process of the zeroth-order convolved orthogonal expansion	S^{*i}	power spectral density (PSD) of ρ^i
Φ_i	fourier transform of ϕ_i	c	correlation of convolved basis processes
$\phi_i^{(n)}$	ith random basis process of the nth-order convolved orthogonal expansion	C	PSD of a convolved correlation
$\Phi_i^{(n)}$	fourier transform of $\phi_i^{(n)}$	g	characteristic kernel of a system
u	output process of a system	G	fourier transform of g
U	Fourier transform of u	g^{*m}	m -fold convolution product of g
r	correlation function	G^m	Fourier transform of g^{*m}
R	spectrum or Fourier transform of r	$K^{(n)}$	n th-order Volterra kernel
ρ	autocorrelation of ϕ_1	$\hat{\phi}_n$	n th-order functional Hermite polynomial
		$\delta(\omega)$	Dirac delta function

demonstrate application of the COE method, particularly its unique feature on the representation of non-Gaussian processes [15].

The paper is organized as follows. In Section 2 the basic properties of the random basis processes (correlation, convolution, and derivatives) are presented in the case of the zeroth-order COE. The case of n th-order COE is examined in Section 3, where the relation to classical Volterra-type expansions is also discussed. The properties of random basis processes are numbered and appended with S and E to indicate specific applicability to stationary and ergodic processes, respectively, e.g., Property 2.3S. It is worth noting that those properties for stationary processes will equally apply to ergodic ones but not vice versa. Some applications of the proposed approach to linear and weakly nonlinear oscillators are presented in Section 4. The paper closes with some concluding remarks in Section 5.

2. THE ZERO-ORDER CONVOLVED ORTHOGONAL EXPANSION

2.1 Correlation of Random Basis Processes

An input Gaussian process $\phi_1(t, \vartheta)$, stationary or nonstationary, is characterized with the autocorrelation function $\rho(t_1, t_2)$ and unit variance, where $\vartheta \in \Theta$ indicates a sample point in random space. Based on the so-called diagonal class of random processes [16], an output process can be represented as the zeroth-order convolved (or memoryless) orthogonal expansion of $\phi_1(t, \vartheta)$ [1],

$$u(t, \vartheta) = \sum_{i=0}^{\infty} u_i(t) \phi_i(t, \vartheta) \quad (1)$$

where the random basis process $\phi_i(t, \vartheta)$ corresponds to the i th-degree Hermite polynomial about $\phi_1(t, \vartheta)$ with $\phi_0(t, \vartheta) = 1$, e.g., $\phi_2(t, \vartheta) = \phi_1^2(t, \vartheta) - 1$, $\phi_3(t, \vartheta) = \phi_1^3(t, \vartheta) - 3\phi_1(t, \vartheta)$, etc. According to the generalized Mehler's formula [17], the n -point correlation of n random basis processes is given as follows:

Property 1.0 [17]

$$\begin{aligned} r_{a_1 a_2 \dots a_n}(t_1, t_2, \dots, t_n) &= \overline{\phi_{a_1}(t_1, \vartheta) \dots \phi_{a_n}(t_n, \vartheta)} \\ &= a_1! \dots a_n! \sum_{\nu_{12}=0}^{\infty} \dots \sum_{\nu_{n-1, n}=0}^{\infty} \delta_{a_1 b_1} \dots \delta_{a_n b_n} \prod_{j < k} \frac{\rho^{\nu_{jk}}(t_j, t_k)}{\nu_{jk}!} \end{aligned} \quad (2a)$$

where the overbar denotes ensemble average, and

$$b_k = \sum_{j \neq k} \nu_{jk}, \quad \nu_{jk} = \nu_{kj}, \quad \delta_{a_k b_k} = \begin{cases} 1 & a_k = b_k \\ 0 & a_k \neq b_k \end{cases} \quad (2b)$$

Following Eq. (2), the two-point and three-point correlation functions are specifically obtained as

Property 1.1 [17]

$$r_{ij}(t_1, t_2) = \overline{\Phi_i(t_1, \vartheta)\Phi_j(t_2, \vartheta)} = \delta_{ij}i!\rho^i(t_1, t_2) \tag{3}$$

Property 1.2 [17]

$$r_{ijk}(t_1, t_2, t_3) = \overline{\Phi_i(t_1, \vartheta)\Phi_j(t_2, \vartheta)\Phi_k(t_3, \vartheta)} = \frac{i!j!k!}{i'!j'!k'!}\rho^{k'}(t_1, t_2)\rho^{j'}(t_1, t_3)\rho^{i'}(t_2, t_3) \tag{4a}$$

$$i' = \frac{j+k-i}{2}, \quad j' = \frac{i+k-j}{2}, \quad k' = \frac{i+j-k}{2} \tag{4b}$$

where i', j', k' must be non-negative integers, otherwise, $r_{ijk} = 0$.

Remark: When the input Gaussian process $\Phi_1(t, \vartheta)$ is stationary, the autocorrelation expression $\rho(t_1, t_2)$ in Eqs. (2)–(4) and following throughout the paper can be simply written as $\rho(t_1 - t_2)$.

Denote \mathcal{F} the Fourier transform operator, and let $\Phi(\omega) = \mathcal{F}[\varphi(t)]$, $R(\omega) = \mathcal{F}[r(t)]$, and $S^{*i}(\omega) = \mathcal{F}[\rho^i(t)]$, where we assume that the Fourier transform of the random basis processes always exists. For ergodic processes, with the tilde denoting a complex conjugate, the power spectral density (PSD) is obtained as

Property 1.1E

$$R_{ij}(\omega) = \delta_{ij}i!S^{*i}(\omega) \tag{5a}$$

$$S^{*i}(\omega) = \frac{1}{T}|\Phi_i(\omega, \vartheta)|^2 \tag{5b}$$

Proof: For an ergodic process, the ensemble average in Eq. (3) can be replaced by the time average

$$r_{ij}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \Phi_i(\tau + t_2, \vartheta)\Phi_j(t_2, \vartheta)dt_2 = \frac{1}{T}\Phi_i(t, \vartheta) * \Phi_j(-t, \vartheta) = \delta_{ij}i!\frac{1}{T}\Phi_i(t, \vartheta) * \Phi_i(-t, \vartheta) \tag{6a}$$

where $\tau = t_1 - t_2$ and the symbol $*$ denotes the convolution operator. In the frequency domain, (6a) becomes

$$R_{ij}(\omega) = \delta_{ij}i!\frac{1}{T}\Phi_i(\omega, \vartheta)\Phi_i(-\omega, \vartheta) = \delta_{ij}i!\frac{1}{T}\Phi_i(\omega, \vartheta)\tilde{\Phi}_i(\omega, \vartheta) = \delta_{ij}i!\frac{1}{T}|\Phi_i(\omega, \vartheta)|^2 \tag{6b}$$

2.2 Convolution of Random Basis Processes

The convolution functions of the random basis processes are useful in the convolution-type stochastic variational method [14]. The n -point convolution of the first n random basis processes is given as

Property 2.0

$$\begin{aligned} c_{a_1 a_2 \dots a_n}(t) &= \overline{\Phi_{a_1}(t, \vartheta) * \Phi_{a_2}(t, \vartheta) * \dots * \Phi_{a_n}(t, \vartheta)} \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \overline{\Phi_{a_1}(t - t_2 - \dots - t_n, \vartheta)\Phi_{a_2}(t_2, \vartheta) \dots \Phi_{a_n}(t_n, \vartheta)} dt_2 \dots dt_n \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} r_{a_1 a_2 \dots a_n}(t_1, t_2, \dots, t_n) dt_2 \dots dt_n \\ &= a_1! \dots a_n! \sum_{\nu_{12}=0}^{\infty} \dots \sum_{\nu_{n-1,n}=0}^{\infty} \delta_{a_1 b_1} \dots \delta_{a_n b_n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{j < k} \frac{1}{\nu_{jk}!} \rho^{\nu_{jk}}(t_j, t_k) dt_2 \dots dt_n \\ &\qquad\qquad\qquad t_1 = t - t_2 - t_3 - \dots - t_n \end{aligned} \tag{7}$$

and the variables refer to Eq. (2b). The two-point and three-point convolution functions are specifically obtained as

Property 2.1

$$c_{ij}(t) = \overline{\Phi_i(t, \vartheta) * \Phi_j(t, \vartheta)} = \delta_{ij} i! \int_{-\infty}^{\infty} \rho^i(t - t_2, t_2) dt_2 \quad (8)$$

Property 2.2

$$c_{ijk}(t) = \overline{\Phi_i(t, \vartheta) * \Phi_j(t, \vartheta) * \Phi_k(t, \vartheta)} = \frac{i!j!k!}{i'!j'!k'!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho^{k'}(t - t_2 - t_3, t_2) \rho^{j'}(t - t_2 - t_3, t_3) \rho^{i'}(t_2, t_3) dt_2 dt_3 \quad (9)$$

where i' , j' , and k' refer to (4b).

For the stationary case, Eq. (8) reduces to

Property 2.1S

$$c_{ij}(t) = \delta_{ij} i! \int_{-\infty}^{\infty} \rho^i(t - 2t_2) dt_2 = \delta_{ij} i! \tau_i \quad (10)$$

where $\tau_i = \int_0^{\infty} \rho^i(t) dt$ is the correlation time of the i th random basis process $\Phi_i(t, \vartheta)$.

The two-point convolution functions, i.e., Eqs. (8) and (10), can be rewritten in the frequency domain for the nonstationary and stationary cases, respectively, as

$$C_{ij}(\omega) = \overline{\Phi_i(\omega, \vartheta) \Phi_j(\omega, \vartheta)} = \delta_{ij} i! \mathcal{F} \left(\int_{-\infty}^{\infty} \rho^i(t - t_2, t_2) dt_2 \right) \quad (8b)$$

$$C_{ij}(\omega) = \overline{\Phi_i(\omega, \vartheta) \Phi_j(\omega, \vartheta)} = 2\pi \delta_{ij} i! \tau_i \delta(\omega) \quad (10b)$$

where $C(\omega) = \mathcal{F}(c(t))$. Similarly the three-point convolution function in the frequency domain is obtained as

$$\begin{aligned} C_{ijk}(\omega) &= \overline{\Phi_i(\omega, \vartheta) \Phi_j(\omega, \vartheta) \Phi_k(\omega, \vartheta)} \\ &= \frac{i!j!k!}{i'!j'!k'!} \mathcal{F} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho^{k'}(t - t_2 - t_3, t_2) \rho^{j'}(t - t_2 - t_3, t_3) \rho^{i'}(t_2, t_3) dt_2 dt_3 \right) \end{aligned} \quad (11)$$

where i' , j' , and k' refer to (4b). For the stationary case, (11) reduces to

Property 2.2S

$$C_{ijk}(\omega) = \delta(\omega) \frac{1}{3} \frac{i!j!k!}{i'!j'!k'!} \int_{-\infty}^{\infty} S^{*k'}(\omega') S^{*j'}(\omega') S^{*i'}(\omega') d\omega' \quad (12)$$

Proof:

$$\begin{aligned} C_{ijk}(\omega) &= \overline{\Phi_i(\omega, \vartheta) \Phi_j(\omega, \vartheta) \Phi_k(\omega, \vartheta)} \\ &= \frac{i!j!k!}{i'!j'!k'!} \mathcal{F} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho^{k'}(t - 2t_2 - t_3) \rho^{j'}(t - t_2 - 2t_3) \rho^{i'}(t_2 - t_3) dt_2 dt_3 \right) \end{aligned} \quad (13)$$

Let $\tau = t - 2t_2 - t_3$ and $\tau' = t - t_2 - 2t_3$. Equation (13) then becomes

$$C_{ijk}(\omega) = \frac{i!j!k!}{i'!j'!k'!} \mathcal{F} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho^{k'}(\tau) \rho^{j'}(\tau') \rho^{i'}(\tau' - \tau) |J| d\tau d\tau' \right) \quad (14)$$

with the determinant of the Jacobian matrix

$$|J| = \begin{vmatrix} \frac{\partial t_2}{\partial \tau} & \frac{\partial t_2}{\partial \tau'} \\ \frac{\partial t_3}{\partial \tau} & \frac{\partial t_3}{\partial \tau'} \end{vmatrix} = \begin{vmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \frac{1}{3} \tag{15}$$

Equation (14) therefore immediately leads to Eq. (12).

2.3 Derivatives of Random Basis Processes

The derivatives of random basis processes are useful when the derivatives of the output process (1) have physical meaning, e.g., velocity and acceleration obtained from displacement. The n -point correlation of the derivatives of the first n random basis processes is given as

Property 3.0

$$\begin{aligned} r_{a_1 \dots a_n, d_1 \dots d_n}(t_1, \dots, t_n) &= \overline{\Phi_{a_1, d_1}(t_1, \vartheta) \dots \Phi_{a_n, d_n}(t_n, \vartheta)} \\ &= a_1! \dots a_n! \sum_{\nu_{12}=0}^{\infty} \dots \sum_{\nu_{n-1, n}=0}^{\infty} \delta_{a_1 b_1} \dots \delta_{a_n b_n} \frac{\partial^{d_1 + \dots + d_n}}{\partial t_1^{d_1} \dots \partial t_n^{d_n}} \prod_{j < k} \frac{\rho^{\nu_{jk}}(t_j, t_k)}{\nu_{jk}!} \end{aligned} \tag{16}$$

where the subscripts d_i denote d_i th derivatives about t_i . The two- and three-point correlations of the derivatives are explicitly given as

Property 3.1

$$r_{ij, pq}(t_1, t_2) = \overline{\Phi_{i, p}(t_1, \vartheta) \Phi_{j, q}(t_2, \vartheta)} = \delta_{ij} i! \frac{\partial^{p+q}}{\partial t_1^p \partial t_2^q} \rho^i(t_1, t_2) \tag{17}$$

Property 3.2

$$r_{ijk, pqv}(t_1, t_2, t_3) = \overline{\Phi_{i, p}(t_1, \vartheta) \Phi_{j, q}(t_2, \vartheta) \Phi_{k, v}(t_3, \vartheta)} = \frac{i! j! k!}{i! j! k!} \frac{\partial^{p+q+v}}{\partial t_1^p \partial t_2^q \partial t_3^v} \left[\rho^{k'}(t_1, t_2) \rho^{j'}(t_1, t_3) \rho^{i'}(t_2, t_3) \right] \tag{18}$$

where $i', j',$ and k' refer to Eq. (4b). In the stationary case, Eqs. (17) and (18) can be further written, respectively, as

Property 3.1S

$$r_{ij, pq}(\tau) = \delta_{ij} (-1)^q i! \frac{\partial^{p+q}}{\partial \tau^{p+q}} \rho^i(\tau) \tag{19}$$

$\tau = t_1 - t_2$, with $r_{ij, pq}(0) = 0$ when $p + q$ is odd, and

Property 3.2S

$$r_{ijk, pqv}(\tau, \tau') = \frac{i! j! k!}{i! j! k!} (-1)^{q+v} \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau'} \right)^p \frac{\partial^{q+v}}{\partial \tau^q \partial \tau'^v} \left[\rho^{k'}(\tau) \rho^{j'}(\tau') \rho^{i'}(\tau' - \tau) \right] \tag{20}$$

where $\tau = t_1 - t_2$ and $\tau' = t_1 - t_3$.

The n -point convolution of the derivatives is similarly obtained as

Property 4.0

$$\begin{aligned} c_{a_1 \dots a_n, d_1 \dots d_n}(t) &= \overline{\Phi_{a_1, d_1}(t, \vartheta) * \Phi_{a_2, d_2}(t, \vartheta) * \dots * \Phi_{a_n, d_n}(t, \vartheta)} \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \overline{\Phi_{a_1, d_1}(t - t_2 - \dots - t_n, \vartheta) \Phi_{a_2, d_2}(t_2, \vartheta) \dots \Phi_{a_n, d_n}(t_n, \vartheta)} dt_2 \dots dt_n \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} r_{a_1 \dots a_n, d_1 \dots d_n}(t_1, t_2, \dots, t_n) dt_2 \dots dt_n \\ &= a_1! \dots a_n! \sum_{\nu_{12}=0}^{\infty} \dots \sum_{\nu_{n-1, n}=0}^{\infty} \delta_{a_1 b_1} \dots \delta_{a_n b_n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\partial^{d_1 + \dots + d_n}}{\partial t_1^{d_1} \dots \partial t_n^{d_n}} \prod_{j < k} \frac{1}{\nu_{jk}!} \rho^{\nu_{jk}}(t_j, t_k) dt_2 \dots dt_n \end{aligned} \tag{21}$$

$$t_1 = t - t_2 - t_3 - \dots - t_n$$

The two-point convolution of the derivatives specifically has the following property:

Property 4.1

$$C_{ij,pq}(\omega) = \overline{(\sqrt{-1}\omega)^{p+q} \Phi_i(\omega, \vartheta) \Phi_j(\omega, \vartheta)} = \begin{cases} 0 & p \neq 0 \text{ or } q \neq 0 \\ 0 & i \neq j \text{ \& } p = q = 0 \\ \infty & i = j = p = q = 0 \end{cases} \quad (22)$$

Proof: The property can be proved in the time domain, i.e.,

$$c_{ij,pq}(t) = \overline{\Phi_{i,p}(t, \vartheta) * \Phi_{j,q}(t, \vartheta)} = \delta_{ij} i! \int_{-\infty}^{\infty} \frac{\partial^{p+q}}{\partial t_1^p \partial t_2^q} \rho^i(t_1, t_2) dt_2 \quad (23)$$

where $t_1 = t - t_2$. Considering $\rho(\infty, -\infty) = 0$ and the symmetry of the convolution operator, (23) directly yields the results in Eq. (22).

Similarly, we have the following property for the three-point convolution of the derivatives:

Property 4.2

$$C_{ijk,pqv}(\omega) = \overline{(\sqrt{-1}\omega)^{p+q+v} \Phi_i(\omega, \vartheta) \Phi_j(\omega, \vartheta) \Phi_k(\omega, \vartheta)} = 0 \text{ when } p \neq 0, q \neq 0, \text{ or } v \neq 0 \quad (24)$$

Proof: In the time domain

$$\begin{aligned} c_{ijk,pqv}(t) &= \overline{\Phi_{i,p}(t, \vartheta) * \Phi_{j,q}(t, \vartheta) * \Phi_{k,v}(t, \vartheta)} \\ &= \frac{i!j!k!}{i'!j'!k'!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^{p+q+v}}{\partial t_1^p \partial t_2^q \partial t_3^v} \left[\rho^{k'}(t_1, t_2) \rho^{j'}(t_1, t_3) \rho^{i'}(t_2, t_3) \right] dt_3 dt_2 \end{aligned} \quad (25)$$

where $t_1 = t - t_2 - t_3$. Considering $\rho(t_1, \pm\infty) = 0$ for any finite t_1 and the symmetry of the convolution operator, Eq. (25) leads to Eq. (24).

3. THE N TH-ORDER CONVOLVED ORTHOGONAL EXPANSION

3.1 A New Expansion Beyond Classical Volterra-Type Expansions

To model nonlinear systems, we propose to generalize the zeroth-order or memoryless orthogonal expansion presented above to an n th-order COE:

$$u(t, \vartheta) = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} u_i^{(n)}(t) \Phi_i^{(n)}(t, \vartheta) \quad (26)$$

$$\Phi_i^{(n)}(t, \vartheta) = \overbrace{g * g * \dots * g}^n * \Phi_i = g^{*n} * \Phi_i \quad (27)$$

where $g(t, t')$ is a given kernel, typically a Green's function. For a stationary kernel $g(t - t')$, Eq. (27) can be simplified in the frequency domain as

$$\Phi_i^{(n)}(\omega, \vartheta) = \overbrace{G(\omega)G(\omega) \dots G(\omega)}^n \Phi_i(\omega, \vartheta) = G^n(\omega) \Phi_i(\omega, \vartheta) \quad (28)$$

where $G(\omega) = \mathcal{F}[g(t)]$. For notational simplicity, the superscript (0) for the zeroth-order COE is usually omitted throughout the paper. The memoryless orthogonal expansion (1) thus corresponds to the zeroth-order COE with $n = 0$ in (26).

The advantage of the n th-order COE (26) can be especially demonstrated by comparing it with classical Volterra-type expansions. The general Volterra series expansion about a stationary input process $\phi_1(t, \vartheta)$ is written as [18]

$$u(t, \vartheta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K^{(n)}(t_1, t_2 \cdots t_n) \phi_1(t - t_1, \vartheta) \phi_1(t - t_2, \vartheta) \cdots \phi_1(t - t_n, \vartheta) dt_1 dt_2 \cdots dt_n \quad (29)$$

Based on Cameron and Martin’s result [19], a variant of (29) was developed as [20, 21]

$$u(t, \vartheta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K^{(n)}(t_1, t_2 \cdots t_n) \hat{\phi}_n(t - t_1, t - t_2, \cdots t - t_n, \vartheta) dt_1 dt_2 \cdots dt_n \quad (30)$$

where $\hat{\phi}_n$ is the n th-order functional Hermite polynomial, e.g., $\hat{\phi}_2(t_1, t_2, \vartheta) = \phi_1(t_1, \vartheta)\phi_1(t_2, \vartheta) - \delta(t_1 - t_2)$. A further specialization of (30) can be given as [21]

$$u(t, \vartheta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} K^{(n)}(t - t_1) \phi_n(t_1, \vartheta) dt_1 \quad (31)$$

which is close to the COE representation, Eq. (26), except the kernels in Eq. (31) are unknown. Indeed, a common issue of all the Volterra-type representations is severe difficulties in solving the unknown kernels $K^{(n)}$. In the COE representation, all the kernels are explicitly given and the problem is significantly reduced to determine the unknown coefficients $u_i^{(n)}$. Some properties of convolved random basis processes are presented below.

3.2 Properties of Convolved Random Basis Processes

The n -point correlation of the convolved random basis processes is therefore obtained as

Property 5.0

$$\begin{aligned} r_{a_1 \cdots a_n}^{m_1 \cdots m_n}(t_1, t_2, \cdots, t_n) &= \overline{\phi_{a_1}^{(m_1)}(t_1, \vartheta) \cdots \phi_{a_n}^{(m_n)}(t_n, \vartheta)} \\ &= a_1! \cdots a_n! \sum_{\nu_{12}=0}^{\infty} \cdots \sum_{\nu_{n-1,n}=0}^{\infty} \delta_{a_1 b_1} \cdots \delta_{a_n b_n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g^{*m_1}(t_1, t'_1) \cdots \\ &g^{*m_n}(t_n, t'_n) \prod_{j < k} \frac{\rho^{\nu_{jk}}(t_j, t_k)}{\nu_{jk}!} dt'_1 \cdots dt'_n \end{aligned} \quad (32)$$

with the two- and three-point correlation functions explicitly given as

Property 5.1

$$r_{ij}^{mn}(t_1, t_2) = \overline{\phi_i^{(m)}(t_1, \vartheta) \phi_j^{(n)}(t_2, \vartheta)} = \delta_{ij} i! \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^{*m}(t_1, t'_1) g^{*n}(t_2, t'_2) \rho^i(t'_1, t'_2) dt'_1 dt'_2 \quad (33)$$

Property 5.2

$$\begin{aligned} r_{ijk}^{mnl}(t_1, t_2, t_3) &= \overline{\phi_i^{(m)}(t_1, \vartheta) \phi_j^{(n)}(t_2, \vartheta) \phi_k^{(l)}(t_3, \vartheta)} \\ &= \frac{i! j! k!}{i'! j'! k'!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^{*m}(t_1, t'_1) g^{*n}(t_2, t'_2) g^{*l}(t_3, t'_3) \rho^{k'}(t'_1, t'_2) \rho^{j'}(t'_1, t'_3) \rho^{i'}(t'_2, t'_3) dt'_1 dt'_2 dt'_3 \end{aligned} \quad (34)$$

where i', j' , and k' refer to (4b).

In the stationary case, i.e., both the kernel and the input of a system being stationary, the two-point correlation is simplified in the frequency domain as follows:

Property 5.1S

$$R_{ij}^{mn}(\omega) = \delta_{ij} i! G^m(\omega) \tilde{G}^n(\omega) S^{*i}(\omega) \quad (35)$$

Proof: In the stationary case (33) reduces to

$$r_{ij}^{mn}(t_1 - t_2) = \delta_{ij} i! \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^{*m}(t_1 - t'_1) g^{*n}(t_2 - t'_2) \rho^i(t'_1 - t'_2) dt'_1 dt'_2 \quad (36)$$

Let $\tau = t_1 - t'_1$, $\tau' = t_2 - t'_2$, and $\Delta\tau = t_2 - t_1$. Equation (36) then becomes

$$r_{ij}^{mn}(\Delta\tau) = \delta_{ij} i! \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^{*m}(\tau) g^{*n}(\tau') \rho^i(\tau' - \tau - \Delta\tau) d\tau d\tau' \quad (37)$$

and the Fourier transform directly leads to (35).

One notes when $m = n$, the PSD in (35) is positive, consistent with the positive definiteness of an autocorrelation function.

For ergodic processes, following Eq. (5), Eq. (35) can be also written as

Property 5.1E

$$R_{ij}^{mn}(\omega) = \delta_{ij} \frac{1}{T} \Phi_i^{(m)}(\omega, \vartheta) \tilde{\Phi}_i^{(n)}(\omega, \vartheta) \quad (38)$$

Similarly the n -point correlation of the derivatives of the convolved random basis processes is obtained as

Property 6.0

$$\begin{aligned} r_{a_1 \dots a_n, d_1 \dots d_n}^{m_1 \dots m_n}(t_1, t_2, \dots, t_n) &= \overline{\Phi_{a_1, d_1}^{(m_1)}(t_1, \vartheta) \dots \Phi_{a_n, d_n}^{(m_n)}(t_n, \vartheta)} \\ &= a_1! \dots a_n! \sum_{\nu_{12}=0}^{\infty} \dots \sum_{\nu_{n-1, n}=0}^{\infty} \delta_{a_1 b_1} \dots \delta_{a_n b_n} \frac{\partial^{d_1 + \dots + d_n}}{\partial t_1^{d_1} \dots \partial t_n^{d_n}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g^{*m_1}(t_1, t'_1) \dots \\ &g^{*m_n}(t_n, t'_n) \prod_{j < k} \frac{\rho^{\nu_{jk}}(t_j, t_k)}{\nu_{jk}!} dt'_1 \dots dt'_n \end{aligned} \quad (39)$$

with the two- and three-point correlations explicitly given as

Property 6.1

$$r_{ij, pq}^{mn}(t_1, t_2) = \overline{\Phi_{i, p}^{(m)}(t_1, \vartheta) \Phi_{j, q}^{(n)}(t_2, \vartheta)} = \delta_{ij} i! \frac{\partial^{p+q}}{\partial t_1^p \partial t_2^q} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^{*m}(t_1, t'_1) g^{*n}(t_2, t'_2) \rho^i(t'_1, t'_2) dt'_1 dt'_2 \quad (40)$$

Property 6.2

$$\begin{aligned} r_{ijk, pqv}^{mnl}(t_1, t_2) &= \overline{\Phi_{i, p}^{(m)}(t_1, \vartheta) \Phi_{j, q}^{(n)}(t_2, \vartheta) \Phi_{k, v}^{(l)}(t_3, \vartheta)} \\ &= \frac{i! j! k!}{i! j! k!} \frac{\partial^{p+q+v}}{\partial t_1^p \partial t_2^q \partial t_3^v} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^{*m}(t_1, t'_1) g^{*n}(t_2, t'_2) g^{*l}(t_3, t'_3) \rho^{k'}(t'_1, t'_2) \rho^{j'}(t'_1, t'_3) \rho^{i'}(t'_2, t'_3) dt'_1 dt'_2 dt'_3 \end{aligned} \quad (41)$$

In the stationary case, the two-point correlation is simplified in frequency domain as

Property 6.1S

$$R_{ij, pq}^{mn}(\omega) = \delta_{ij} i! (\sqrt{-1}\omega)^{p+q} G^m(\omega) \tilde{G}^n(\omega) S^{*i}(\omega) \quad (42)$$

which is directly derived from (35).

4. APPLICATION EXAMPLES OF CONVOLVED ORTHOGONAL EXPANSIONS

4.1 Linear Oscillators

The application of COE to linear oscillators first appeared in [1]. Below the idea is succinctly presented, which also prepares for the following example on nonlinear oscillators. Suppose the linear oscillator

$$\begin{aligned} \ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2u &= f \\ u(0) = \dot{u}(0) &= 0 \end{aligned} \tag{43}$$

is subjected to a nonstationary non-Gaussian translation process input, i.e.,

$$f(t, \vartheta) = \sum_{i=0} f_i(t)\Phi_i(t, \vartheta) \tag{44}$$

By using the Green's function

$$g(t) = \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{45}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{46}$$

$$G(\omega) = \frac{1}{\omega_n^2 - \omega^2 + \sqrt{-1}2\zeta\omega\omega_n} \tag{47}$$

the first three correlations of the nonstationary output u can be directly calculated from

$$\bar{u}(t) = \int_0^t g(t - \tau) f_0(\tau) d\tau \tag{48}$$

$$r_{uu}(t_1, t_2) = \int_0^{t_2} \int_0^{t_1} g(t_1 - \tau_1) g(t_2 - \tau_2) \sum_{i=0} i! \rho^i(\tau_1 - \tau_2) f_i(\tau_1) f_i(\tau_2) d\tau_1 d\tau_2 \tag{49}$$

$$r_{uuu}(t_1, t_2, t_3) = \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} g(t_1 - \tau_1) g(t_2 - \tau_2) g(t_3 - \tau_3) \sum_{i,j,k=0} r_{ijk}(\tau_1 - \tau_2, \tau_1 - \tau_3, \tau_2 - \tau_3) d\tau_1 d\tau_2 d\tau_3 \tag{50}$$

with r_{ijk} given in Eq. (4).

When the excitation in Eq. (43) is stationary, the output process is directly given as

$$u(t, \vartheta) = \sum_{i=0} f_i \Phi_i^{(1)}(t, \vartheta) \tag{51}$$

which is a special case of the COE representation (26), i.e., the first-order COE. Note that given the Green's function g and the underlying Gaussian process, the stationary probability density function (pdf) of the output in Eq. (51) can be rapidly estimated by using Monte Carlo method in the frequency domain.

A numerical example for application of the COE on linear oscillator is given in [1]. With regard to the multi-degree-of-freedom (MDOF) linear systems, the oscillator equations given above can be directly applied by using the modal decomposition as shown in [22].

4.2 Weakly Nonlinear Oscillators

The accurate computation of the response of nonlinear single-degree-of-freedom (SDOF) oscillators under stochastic loading is important in earthquake engineering where equivalent nonlinear SDOF systems are often used in order to avoid the computationally intensive nonlinear response history analysis of MDOF systems. These equivalent systems retain some of the dynamic characteristics of the real structure and are capable of representing certain response quantities of the MDOF structure (see, e.g., Ref. [23]).

In this example a Duffing oscillator subjected to a Gaussian white noise excitation with intensity D is considered:

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2(u + \alpha u^3) = W \quad (52)$$

Similar to the idea presented in [1, 14, 22] on multiscale decomposition of random field problems, Eq. (52) can be decomposed into a reference linear filter

$$\ddot{u}_0 + 2\zeta\omega_n\dot{u}_0 + \omega_n^2 u_0 = W \quad (53)$$

and a nonlinear fluctuation problem

$$\ddot{u}' + 2\zeta\omega_n\dot{u}' + \omega_n^2 u' = -\alpha\omega_n^2(u_0 + u')^3 \quad (54)$$

with

$$u = u_0 + u' \quad (55)$$

The Gaussian response of the linear filter (53) is straightforwardly obtained as

$$u_0(t, \vartheta) = \sigma_0 \varphi_1(t, \vartheta), \quad (56)$$

where $\varphi_1(t, \vartheta)$ is characterized by unit variance and power spectral density

$$S = \frac{D}{\sigma_0^2} |G(\omega)|^2 \quad (57)$$

and the variance (see e.g., [24])

$$\sigma_0^2 = \frac{1}{2\pi} D \int_{-\infty}^{\infty} \left| \frac{1}{\omega_n^2 - \omega^2 + \sqrt{-1}2\zeta\omega\omega_n} \right|^2 d\omega = \frac{D}{4\zeta\omega_n^3} \quad (58)$$

For a small α , by using iterative substitution the output of the nonlinear fluctuation problem (54) is obtained as

$$u' = -\alpha\omega_n^2\sigma_0^3 g * \varphi_1^3 + 3\alpha^2\omega_n^4\sigma_0^5 g * (\varphi_1^2 * \varphi_1^3) + O(\alpha^3) \quad (59)$$

For Gaussian-based random basis processes, i.e., Hermite polynomials,

$$\varphi_1^3 = \varphi_3 + 3\varphi_1, \quad \varphi_1^2 = \varphi_2 + 1 \quad (60)$$

The sum of Eqs. (56) and (59) can be rewritten in terms of the random basis processes in time and frequency domain, respectively, as

$$u/\sigma_0 = \varphi_1 - \beta \left(\varphi_3^{(1)} + 3\varphi_1^{(1)} \right) + 3\beta g * \left[(\varphi_2 + 1) \left(\beta(\varphi_3^{(1)} + 3\varphi_1^{(1)}) \right) \right] + O(\beta^3 g^{*3}) \quad (61)$$

$$U/\sigma_0 = \Phi_1 - \beta \left(\Phi_3^{(1)} + 3\Phi_1^{(1)} \right) + 3\beta G \left[(\Phi_2 + 2\pi\delta(0)) * \left(\beta(\Phi_3^{(1)} + 3\Phi_1^{(1)}) \right) \right] + O((\beta G)^3) \quad (62)$$

where $\beta = \alpha\sigma_0^2\omega_n^2$.

By invoking ergodicity and using Properties 1.1E, 1.2, 5.1S, and 5.2, the stationary PSD of the output displacement becomes

$$S_{UU} = \frac{1}{T}U\tilde{U} = \sigma_0^2 S \left[1 - 3\beta(G + \tilde{G}) + 3\beta^2 \left(3|G|^2 + 2|G|^2 \frac{S^{*3}}{S} + 3(G^2 + \tilde{G}^2) + 6G(G * S^{*2}) + 6\tilde{G}(\tilde{G} * S^{*2}) + 6(G + \tilde{G}) \langle GS \rangle \right) + O((\beta |G|)^3) \right] \tag{63}$$

where $\langle GS \rangle = \int_{-\infty}^{\infty} G(\omega)S(\omega)d\omega$. Since

$$\int_{-\infty}^{\infty} \frac{2(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} \left| \frac{1}{\omega_n^2 - \omega^2 + \sqrt{-1}2\zeta\omega\omega_n} \right|^2 d\omega = \frac{\pi}{2\zeta\omega_n^5} \tag{64}$$

(see, e.g., [24]), the variance calculated from the first two terms of Eq. (63) is simply obtained as

$$\sigma^2 = \sigma_0^2(1 - 3\alpha\sigma_0^2) \tag{65}$$

which is identical to the result obtained using other approaches, e.g., [25, 26].

To justify the perturbation approach, we need the nonlinearity factor $\beta |G| < 1$, i.e.,

$$\frac{\alpha\sigma_0^2}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} < 1$$

In the low-frequency range where $\omega \ll \omega_n$, it leads to the condition $\alpha\sigma_0^2 < 1$. In the high-frequency range where $\omega \gg \omega_n$, the condition becomes much more relaxed as $\alpha\sigma_0^2(\omega_n/\omega)^2 < 1$. In the intermediate-frequency range, especially near resonance, when $\omega \sim \omega_n$, the weak nonlinearity condition becomes damping controlled, i.e.,

$$\gamma = \frac{\alpha\sigma_0^2}{2\zeta} = \frac{\alpha D}{8\zeta^2\omega_n^3} < 1 \tag{66}$$

In summary, the condition for all frequency ranges is controlled by the nonlinearity factor γ . As an example, choose $\zeta = 0.02$, $\omega_n = 1$, and the response PSD is obtained by using the second-order perturbation approach (62). With the intensity of white noise fixed at $D = 1$, Fig. 1 shows that the PSD shifts to the right and up with increase of the nonlinearity α from 0 ($\gamma = 0$), to 0.00025 ($\gamma = 0.078125$), and to 0.0005 ($\gamma = 0.15625$). When the nonlinearity α is fixed at 0.00025, Fig. 2 shows clearly the rise and the right shift of PSD with increase of the intensity D from 0.5 ($\gamma = 0.039063$) to 1 ($\gamma = 0.078125$) and to 2 ($\gamma = 0.15625$). The result reflects exactly the well-known phenomenon concerning the natural frequency of the single-well Duffing oscillator.

In addition to serving as a verification of the COE method, this example demonstrates the simplicity and efficiency of the convolved orthogonal expansions for nonlinear problems.

5. CONCLUSION

By developing random process-based orthogonal expansions as an alternative way to represent high-dimensional fluctuations in the time domain, the proposed COE method opens a new direction to deal with nonlinear stochastic dynamics. The advantage is especially noted for its potential efficiency in computing of large and nonlinear dynamical systems, in comparison with the classical Volterra-type representation and the random-variable-based polynomial chaos expansions. Future work will be devoted to application of the COE method to strongly nonlinear systems by using the variational method [14, 27], where the properties presented in this work are expected to play an essential role. A final remark is that the fundamentals of the COE are built on the polynomials. To apply the COE to a nonpolynomial nonlinearity, we will need to approximate the particular nonlinearity with polynomials using, e.g., Taylor expansion.

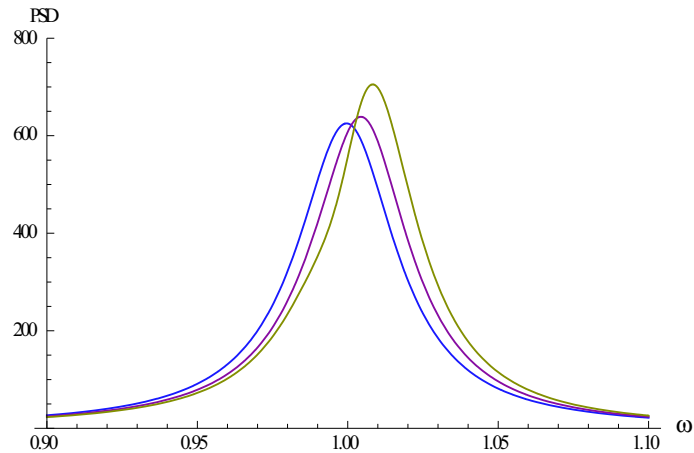


FIG. 1: The PSD of the displacement (from left to right $\alpha = 0, 0.00025,$ and $0.0005,$ respectively) with $\zeta = 0.02,$ $\omega_n = 1,$ and $D = 1.$

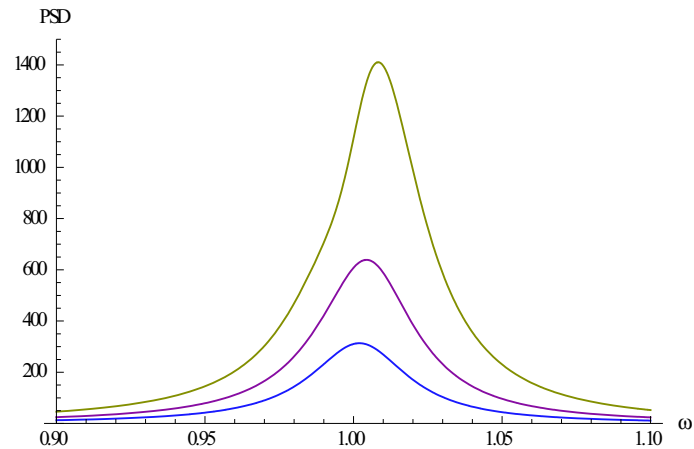


FIG. 2: The PSD of the displacement (from bottom to top $D = 0.5, 1,$ and $2,$ respectively) with $\zeta = 0.02,$ $\omega_n = 1,$ and $\alpha = 0.00025.$

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