INFLUENCE OF INTERNAL ENERGY VARIATIONS OF THE INTERFACE ON THE STABILITY OF FILM FLOW

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To study the mechanisms causing the appearance of spatial structures in a liquid film driven by a gas flux, a novel mathematical model has been proposed. The model is based on the Navier–Stokes and heat conduction equations. We use a condition of the energy balance on the liquid–gas interface. The condition is a consequence of the integral law of conservation and takes into account the energy expended by thermocapillary forces for the interface deformation. The condition describes the formation of the Marangoni stresses caused by the uptake or liberation of heat accompanying the local change of the interfacial surface area. The stability of the film flow and influence of the boundary conditions on the type of the arising instabilities have been studied in the framework of this model. The comparison performed with the regimes observed in the experiments provides the evidence on the qualitative coincidence of the theoretical and experimental results. The solution of the stability problem obtained on the basis of the suggested treatment allows one to predict the formation of the observed self-organizing structures in the experiments.

KEY WORDS: film flow, stability, thermocapillary interface, internal interfacial energy

1. INTRODUCTION

Multiphase flows including liquid non-isothermal media with the deformable liquid–gas interface are widely used in engineering. Modeling these flows is a complex of problems arising in forecasting the dynamics of technological processes (heat transfer enhancement, device cooling, organization of life-support systems, operation of semiconductor display devices, etc.). The first theoretical investigations of the hydrodynamical characteristics of two-phase flows of liquid films and stability of the flows appeared over 50 years ago (Feldman, 1957; Zaitcev, 1960) and later developed in the framework of different analytical approaches (see, for instance, Craik, 1966, 1968; Demekhin, 1976, 1981; Guguchkin et al., 1979; Chan Van Chan and Shkadov, 1979; Jurman and McCready, 1989; Frank, 2006, Kabova et al., 2014a,b, and the bibliography in these papers).

The theoretical problems of hydrodynamics and heat transfer in the film and jet flows with regard to the chemical industry were consistently described in the monograph (Holpanov and Shkadov, 1990). At present, a number of technologies using one and two-phase film flows for the system thermostabilization is increasing due to the development of electronics and manufacturing of complex microelectronic equipment. The falling films are widely used in the technologies of fluidic cooling. The corresponding model equations and the Benard–Marangoni thermocapillary instability were studied in Zeytounian (1998). The influence of the Marangoni forces on the falling film stability was investigated both in the case of uniform heating (Kalliadasis et al., 2003a) and in the presence of the local heater on the
vertical wall (Kalliadasis et al., 2003b). The results of the analytical and experimental investigations of the regimes of the flows depending on the heating intensity and liquid flow rate were presented in Frank and Kabov (2006). The experimental and theoretical evidence as related to the study of the falling films performed in the framework of different approaches was reviewed in Davalos-Orozco (2013). The results of the investigations of the influence of substrate heating conditions, system geometry, mass and thermocapillary forces on the character and type of the arising instabilities in the falling films were described in the review.

The technologies using thin films of the dielectric fluid driven by a co-current gas flux are being developed in the framework of scientific projects supported by the European Space Agency (Celata et al., 2008). Most investigations are concerned with technical aspects connected with the possibility to modify cooling technologies (Kabov et al., 2011; Lyulin and Kabov, 2013, 2014). The theoretical investigations of the influence of different physical and chemical factors (physical properties of the working media, influence of the mass and surface forces) and controlling parameters (gas and liquid flow rates, substrate heating conditions) on the features of the flow regimes are performed in view of the optimization of expensive experiments. The kinematic and thermal characteristics of the structures formed were described in the theoretical studies of the film flows with interfaces (Oron et al., 1997; Sur et al., 2004; Goncharova and Kabov, 2009; Goncharova et al., 2012), taking into consideration the gas and liquid properties (Frank, 2007), including the mechanisms of the film rupture due to the thermocapillary phenomena and substrate heating character (Oron, 2000), as well as owing to the London–van der Waals forces and electrostatic effects (Ajaev, 2013). The combined effect of the thermocapillary and mass forces is taken into account on the basis of the conventional models. However, to fully understand the mechanisms of the structures formation in the film (Liu and Kabov, 2012) and influence of the heat and mass transfer it is required to use new, more complex mathematical models of convection. Thus, taking into account the evaporation under the boundary conditions at the interface allows one to explain the generation of thermal patterns on the surface (Iorio et al., 2011; Goncharova, 2012; Bekezhanova and Goncharova, 2016), including their formation in the presence of the local heat source on the substrate (Gatapova and Kabov, 2008), and to clarify the influence of the boundary conditions on the mechanisms forming a certain flow regime (Goncharova et al., 2013; Goncharova and Rezanova, 2014; Bekezhanova and Goncharova, 2016). The study of the combined effect of the mass transfer at the interface and van der Waals forces on the film dynamics (Shklyaev and Fried, 2007) and stability of the thermocapillary film flows (Bankoff, 1994) allows one to define the character of the impact of each specific factor and to predict the conditions of the flow regime stability. The latter problem is the most important one to further improve the fluidic cooling technologies.

In the stability problem of isothermal states of the media with small viscosity (in particular, cryogenic fluids), taking into account the excess energy and entropy in an interfacial layer allows one to describe the formation mechanism of the surface forces at the interface of the thin liquid films and wave regimes (Andreev et al., 2000). The peculiarity of the mechanism is the finite Marangoni stresses which can arise in the isothermal basic state. In the presence of the external thermal or chemical sources of energy, one is to expect a more substantial effect of this factor on the formation of possible flows regimes of the thin liquid films.

At the moment, there is no general mathematical theory describing the formation of the experimentally observed structures in the film driven by the co-current gas flux (Kabov et al., 2007; Cheverda and Kabov, 2008). The character and structure of the film flow are defined by the combined effect of different factors:

(i) thermocapillary forces,
(ii) mass forces (gravity),
(iii) properties of the liquid and gas phases,
(iv) mass transfer at the interface due to evaporation,
(v) regime of heating the outer boundaries of the system.

Each of them makes a particular contribution.

We focus on (i) to better understand the fundamental aspect of physical phenomena leading to the generation of structures in the films. In the present paper the joint flow of the thin liquid film and co-current gas flux in a minichannel in the gravity field is considered. The Navier–Stokes and heat conductivity equations are used for describing the motion of each phase. The aim of the study is to determine the features of the influence of the energy properties of the thermocapillary interface film–gas on the character of the arising instabilities. For this, the stability problem for the unidirectional stationary flow is solved in the framework of the conjugate boundary problem with the full energy
Influence of Internal Energy Variations of the Interface

The full condition takes into consideration the heat defect in the energy transfer at the interface. The defect characterizes the heat rate (thermal loss) at the interface deformation. The obtained solution is compared with the solution of the analogous problem with the classical condition of the heat flux equality as a condition of the energy balance at the interface. The flow regimes and influence of the internal energy of the interface on the formation of the structures are analyzed.

2. PROBLEM STATEMENT

We consider a non-disturbed flow of the thin liquid film and a uniform laminar flow of gas in the minichannel with the height \( H \). The outer boundaries of the channel are immovable, impermeable solid walls. The liquid and gas fill infinite horizontal layers

\[
\Omega_l = \{ (x, z) : -\infty < x < \infty, -\infty < y < \infty, 0 < z < h(x) \},
\]

\[
\Omega_g = \{ (x, z) : -\infty < x < \infty, -\infty < y < \infty, h(x) < z < H \},
\]

and have a common interface \( \Gamma \) (Fig. 1). We denote \( x = (x, y) \), \( x, y, z \) are the Cartesian coordinates, the values with the indices \( l \) and \( g \) correspond to the liquid and gas, respectively. The stationary motion of the liquid and gas is described by the Navier–Stokes equations and the energy one

\[
\begin{align*}
(v_l \cdot \nabla)v_l & = -\frac{1}{\rho_l} \nabla p_l + \nu_l \Delta v_l + g, \\
\nabla \cdot v_l & = 0, \\
(v_l \cdot \nabla)T_l & = \chi_l \Delta T_l, \\
(x, z) & \in \Omega_l,
\end{align*}
\]

\[
\begin{align*}
(v_g \cdot \nabla)v_g & = -\frac{1}{\rho_g} \nabla p_g + \nu_g \Delta v_g + g, \\
\nabla \cdot v_g & = 0, \\
(v_g \cdot \nabla)T_g & = \chi_g \Delta T_g, \\
(x, z) & \in \Omega_g.
\end{align*}
\]

Here, \( v = (u, w) \) is the velocity vector, \( u = (u, v) \), \( p \) is the pressure, \( T \) is the temperature, \( \rho, \nu, \chi \) are the constant density, kinematic viscosity, and heat diffusivity, respectively, \( \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \) is the gradient, \( \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \) is the Laplace operator. We assume the gas to be an incompressible medium (\( \rho = \text{const} \)) since the system size and the velocities of the media motion are sufficiently small.

**Remark 1.** The channel height in the experiments is \( H = 1.4 \text{ mm} \), and the film rupture is not observed if the velocity of the gas motion \( v < 5 \text{ m/s} \) (Kabov et al., 2011). For the considered flows the Mach number satisfies the condition \( M \ll 1 \) for relative accuracy in the range of 1% due to the neglect of the gas compressibility (Loicyanskii, 2003).

On the solid walls \( z = 0 \) and \( z = H \) the temperature and no-slip conditions are to be valid

\[
\begin{align*}
T_l(x, 0) & = T_{l1}y + T_{l2}, \\
T_g(x, H) & = T_{g1}y + T_{g2}, \\
v_l(x, 0) & = 0, \\
v_g(x, H) & = 0.
\end{align*}
\]

**FIG. 1:** Flow scheme
The tangential forces act along the interface $\Gamma \ (z = h(x))$ and the surface tension $\sigma$ linearly depends on the temperature
\[
\sigma = \sigma_0 - \alpha(T - T_0), \quad \sigma_0, \ \alpha = \text{const}, \quad \sigma_0, \ \alpha > 0. \tag{4}
\]
Here, $\sigma_0, \ T_0$ are the characteristic values of the surface tension and fluid temperature, $\alpha$ is the temperature coefficient of the surface tension.

The conditions of the continuity of the velocities and temperature are imposed at the interface $\Gamma$
\[
v_l(x, h(x)) = v_g(x, h(x)), \quad T_l(x, h(x)) = T_g(x, h(x)) \tag{5}
\]
and the kinematic condition is to be fulfilled
\[
v_l \cdot n = V_n, \tag{6}
\]
where $n = (n_1, n_2, n_3)$ is the unit vector of the external normal to $\Gamma$, $V_n$ is the velocity of the interface $\Gamma$ in the normal direction.

Due to the linear temperature dependence of the surface tension, the energy condition at $\Gamma$ has the form
\[
k_g \frac{\partial T_g}{\partial n}(x, h(x)) - k_l \frac{\partial T_l}{\partial n}(x, h(x)) = \alpha T \nabla \Gamma \cdot v. \tag{7}
\]
Here, $k$ is the heat conductivity coefficient, $\nabla \Gamma$ is the vector differential operator $\nabla \Gamma = \nabla - n(n \cdot \nabla)$, which denotes the surface gradient, $v$ and $T$ are the values of the velocity vector and temperature of both media at $\Gamma$, which pairwise coincide according to (5), then $\nabla \Gamma \cdot v$ is the surface divergence of the vector $v$ at $\Gamma$ $\text{div}_\Gamma v$. The “surface divergence” operator $\text{div}_\Gamma$ acts on the vector function $b = (b_1, b_2, b_3)$, given at $\Gamma$, following the rule
\[
\text{div}_\Gamma b = \nabla \Gamma \cdot b = \sum_{i=1}^{3} \left( \frac{\partial}{\partial x_i} - n_i(n \cdot \nabla) \right) b_i,
\]
where $x_1 \equiv x, \ x_2 \equiv y, \ x_3 \equiv z$.

Condition (7) is a consequence of the integral law of energy conservation and contains non-zero terms in the right hand side of the equation. The derivation of this condition is adduced in the Appendix according to Andreev et al. (2012). A strong derivation of (7) was deduced in Pukhnachov (1989) for the first time. The results of the cited paper are the generalization of the results of Napolitano (1979). The detailed analysis of this condition will be presented in Section 3.

The dynamic condition sets the balance of the forces at the interface (we refer to Edwards et al., 1991; Andreev et al., 2012 for more details)
\[
(P_l - P_g)n = 2\sigma K n + \nabla \Gamma \sigma, \tag{8}
\]
where $P = -pI + 2\mu D(v)$ is the stress tensor in the medium, $D(v) = (\nabla v + (\nabla v)^*)/2$ is the velocity–strain tensor of the vector field $v$, $\mu$ is the coefficient of dynamic viscosity, $K$ is the mean curvature of $\Gamma$. If the surface $\Gamma$ is bent outwards, then $K > 0$.

Except for conditions (3), (5)–(8) the initial thickness of the film is known:
\[
h(-\infty) = h_0. \tag{9}
\]

In the experiments $h_0 \approx 50–100 \ \mu m$ (Cheverda and Kabov, 2008).
To close the problem the mass flow rate of the liquid is set to be
\[
m_l = \int_0^{h_0} \rho_l v_l(z) \, dz.
\]
To construct an exact solution and to realize the numerical computation the boundary conditions have to be rewritten in terms of $h$. For this purpose the appropriate formulae of differential geometry will be used. The surface
\( \Gamma = \{ z = h(x) \} \) is projected uniquely on the lower boundary plane. The vector of the mass forces has the coordinates \( \mathbf{g} = (0, 0, -g) \). The parametrization of \( \Gamma \) is given by mapping \((x, y) \rightarrow (x, y, h(x, y))\). Then, the vectors \( \mathbf{e}_1 = (1, 0, h_x) \) and \( \mathbf{e}_2 = (0, 1, h_y) \) form the local basis in the tangential plane to \( \Gamma \). Now,

\[
\mathbf{e}_1' = N^{-2}(1 + h_y^2, -h_x h_y, h_x), \quad \mathbf{e}_2' = N^{-2}(-h_x h_y, 1 + h_x^2, h_y)
\]

are the vectors which form a co-base. The subscripts \( x \) and \( y \) denote partial derivatives with respect to the corresponding variables, \( N = \sqrt{1 + (\nabla h)^2} = \sqrt{1 + h_x^2 + h_y^2} \). Then,

\[
\nabla \Gamma = \mathbf{e}_1' \frac{\partial}{\partial x} + \mathbf{e}_2' \frac{\partial}{\partial y},
\]

(10)

therefore

\[
\mathbf{e}_1 \cdot \nabla \Gamma = \frac{\partial}{\partial x}, \quad \mathbf{e}_2 \cdot \nabla \Gamma = \frac{\partial}{\partial y}.
\]

(11)

The normal vector to the surface has the coordinates

\[
n = N^{-1}(-\nabla h, 1) = N^{-1}(-h_x, -h_y, 1) = \frac{(-h_x, -h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}},
\]

and the operator of the derivative with respect to the normal has the following form:

\[
\frac{\partial}{\partial n} = N^{-1} \left( \frac{\partial}{\partial z} - \nabla h \cdot \nabla \right).
\]

Here and hereinafter, \( \nabla \) is the two-dimensional operator of differentiation with respect to the variables \( x \) and \( y \).

Let \( \mathbf{G} \) be the basic (metric) tensor of the surface \( \Gamma \) with the components

\[
G_{11} = 1 + h_x^2, \quad G_{12} = h_x h_y, \quad G_{22} = 1 + h_y^2,
\]

and, besides, \( \mathbf{G} \) is the Riemann curvature tensor, since \( \det(G_{ij}) = N^2 \). The components of the contravariant tensor are

\[
G^{11} = N^{-2}(1 + h_y^2), \quad G^{12} = -N^{-2}h_x h_y, \quad G^{22} = N^{-2}(1 + h_x^2).
\]

Let \( K_{ij} \, dx_i \, dx_j \) be the second quadratic form. We suppose that \( x_1 = x, \, x_2 = y \), then

\[
K_{11} = N^{-1} h_{xx}, \quad K_{12} = N^{-1} h_{xy}, \quad K_{22} = N^{-1} h_{yy},
\]

and therefore, the sum of the principal curvatures of \( \Gamma \) is defined as

\[
K = K_{ij} G^{ij} = \text{div}(N^{-1} \nabla h) = \frac{(h_{xx} + h_{yy})(1 + h_x^2 + h_y^2) - (h_x^2 + h_y^2)}{(1 + h_x^2 + h_y^2)^{3/2}}. \quad (12)
\]

Taking into account the above-mentioned, kinematic boundary conditions (6) can be written as follows:

\[
u_l h_x + v_l h_y = w_l,
\]

(13)

since \( h_t = 0 \) for the stationary flow and

\[
V_n = \frac{h_t}{\sqrt{1 + h_x^2 + h_y^2}} = 0.
\]
The energy condition (7) takes the form
\[ k_E \left( \frac{\partial T_2}{\partial z} - h_x \frac{\partial T_g}{\partial x} - h_y \frac{\partial T_g}{\partial y} \right) - k_i \left( \frac{\partial T_i}{\partial z} - h_x \frac{\partial T_i}{\partial x} - h_y \frac{\partial T_i}{\partial y} \right) \] \[ = \alpha T [u_x + v_y + w_z - N^{-1}(h_x^2 u_x + h_x h_y u_y - h_x w_x + h_x h_y v_x + h_y v_y - h_y v_z - h_x w_x - h_y w_y + w_z)]. \] (14)

We assume that \( \Gamma \) is a smooth surface. Taking into account the defined parametrization of \( \Gamma \), the vector triple \( n, e_1, e_2 \) forms the local basis (\( e_1, e_2 \) lie in the plane tangential to \( \Gamma \)). Generally speaking, the local basis is not an orthogonal one, therefore, vector relation (8) is equivalent to the following scalar equations:
\[ p_i - p_g + 2(\mu_g D(v_g) - \mu_i D(v_i)) n \cdot n = 2\sigma K, \]
\[ 2(\mu_g D(v_g) - \mu_i D(v_i)) n \cdot e_{1,2} = \nabla \sigma \cdot e_{1,2}. \]

Since the surface tension \( \sigma \) linearly depends on the temperature, the final form of the relations defining the balance of the forces at \( \Gamma \) is
\[ p_i - p_g + N^{-1}(-d_1 h_x - d_2 h_y + d_3) = 2\sigma K, \] (15)
\[ N^{-1}(d_1 + h_x d_3) = -\alpha T_x - \alpha T_z h_x, \] (16)
\[ N^{-1}(d_2 + h_y d_3) = -\alpha T_y - \alpha T_z h_y. \] (17)

In (15)–(17) the values \( \sigma \) and \( K \) are calculated from (4) and (12), respectively, and the following notations are introduced:
\[ d_1 = \mu_g(u_g + w_g) - \mu_i(u_i + w_i) - 2h_x(\mu_g u_g + \mu_i u_i) - h_y(\mu_g u_g + \mu_i u_i) - h_x(\mu_g w_g - \mu_i w_i), \]
\[ d_2 = \mu_g(v_g + w_g) - \mu_i(v_i + w_i) - h_x(\mu_g v_g + \mu_i v_i) - h_y(\mu_g v_g + \mu_i v_i) - 2h_y(\mu_g v_g - \mu_i v_i), \]
\[ d_3 = 2(\mu_g u_g - \mu_i u_i) - h_x(\mu_g u_g + w_g) - h_x(\mu_i u_i + w_i) - h_y(\mu_g v_g + w_g - \mu_i v_i). \]

3. ANALYSIS OF THE ENERGY CONDITION

According to Gibbs (1873), the interface \( \Gamma \) is a special one-parametric (without surfactant the absolute temperature \( T \) is the state parameter) thermodynamic medium which is a transitional layer (surface phase). This layer is characterized by the additive functions of sets of this surface such as entropy, internal energy, free energy, and others. By using the Gibbs method of excessive values, all the variations of thermodynamic values in the transitional layer with regard to the volumetric phase can be attributed to the representative separating surfacial phase with the excessive thermodynamic characteristics.

In accordance with Gibbs, the specific free energy \( E_f \) of the surfacial phase is axiomatically identified with the surface tension coefficient \( \sigma(T) \). Hence, \( E_f \) is the linear function of the temperature [as described in (4)]. The excessive internal energy of the interface \( E_i \) per unit of the interfacial area is defined as the sum of the free energy \( E_f \) of the interface and formation heat \( q_S \) (Adamson, 1976; Frolov, 1988)
\[ E_i = E_f + q_S, \quad q_S = TS_S = -T \frac{dE_f}{dT} = -T \frac{d\sigma}{dT}, \]
where \( S_S \) is the entropy of the unit of interfacial area. If \( \Sigma \) is the surface area, then the surfacial free energy \( E_f \Sigma \) and heat of the interface formation \( q_S \Sigma \) are the energy characteristics of the interface and they are related to the convective motion of the liquid near \( \Gamma \). Under the usual conditions most thermocapillary phenomena are defined only by the change in the surfacial free energy \( E_f \) (Harper et al., 1967). But, the effect of the internal energy variations can be significant for fluids with small viscosities (Torres and Helborzheimer, 1993) or for flows at a microscale.

It is known that the Marangoni stresses arise at the interface under the local free-energy changes of the unit of interfacial area \( \Gamma \). This leads to the motion of the surfacial phase in the domains with higher surfacial density of the free energy. A few mechanisms are specified resulting in the convective motion close to \( \Gamma \).
1. Classical Marangoni instability. The tangential stresses arise due to the temperature or concentration gradients or external actions keeping the system in the non-equilibrium state and leading to the concurrent convective energy transfer from the volumetric phases to the interface and generate a spontaneous increase of fluctuations of the surface tension (Pearson, 1958; Davis, 1987).

2. Instability of the basic thermocapillary flow caused by the external exerted drop of the temperature along \( \Gamma \). In this case the Marangoni stresses only generate the basic flow in the liquid phase (Davis, 1987); therefore, the evolution of perturbations of the interface occurs through the hydrodynamic mechanism rather than the thermal one.

3. Non-uniform distribution of the temperature along \( \Gamma \) through the energy exchange between the volumetric phases and transitional layer amid them (Harper et al., 1967). The temperature gradients along \( \Gamma \) can arise and be supported due to the local increments of the internal energy of the surfacial phase. Condition (7) means that the heat flux jump in the normal line to \( \Gamma \) is compensated by the change of the internal energy of the interface. We have to take into account the linear dependence of \( \sigma \) on \( T \) and the circumstance that the surfacial density of the internal energy is constant over a wide range of temperature changes (Adamson, 1976). Then, one may state that the above-mentioned change of the internal energy of the interface is connected both with the temperature variation (and therefore, with the specific internal energy) and with the area deformation [the additional summand in the right-hand side of (7) represents this change of the interfacial area]. The latter is to be valid owing to the relation for the change of the area unit \( d\Sigma \) of the interface (Stone, 1990)

\[
\frac{d}{dt}d\Sigma = d\Sigma \nabla \cdot \mathbf{v}.
\]

Thus, it follows from condition (7) that if the internal energy of the interface \( E_i \) per unit of the interfacial area remains constant, then the changes of \( E_i \) of the whole interface attributed to the heat absorption or release are conditioned only by the local variations of the area. The mechanism of forming the tangential stresses through variations of the internal energy of the interface does not require the external energy inflow to the system in the thermal or chemical form.

The significance of the processes of heat absorption/release under the local changes of the transitional layer area for the appearance and development of the convective motion close to \( \Gamma \) is characterized by the parameter \( S = S_M/S_v \). Here, \( S_M = \alpha \delta T/h_s, S_v = \nu u_s/h_s \) are the characteristic values of the Marangoni stresses and viscous stresses, respectively, \( \delta T = T_x u_s x/k_i \) is the characteristic temperature drop owing to the local changes of the surfacial internal energy, \( h_s, u_s, T_x \) are the length, velocity and temperature scales, respectively.

The significance of the effect seems to be first investigated in Torres and Helborzheimer (1993). The parametric analysis of condition (7) was performed in Iorio et al. (2011) and Goncharova (2012) for the problem on the joint flow of the film and co-current gas flux in the presence of evaporation. The order of the additional summand was shown to be comparable with the order of other significant criteria for the film flow in mini- and microchannels. The results of the investigation of the equilibrium stability in a two-layer system with solid walls with regard to the changes of the internal energy of the interface were presented in Andreev et al. (2000). For liquids with small viscosity the variations of \( E_i \) proved to have the stabilizing influence on the Benard–Marangoni instability development.

To estimate the impact of the internal energy variations on the character and type of possible instabilities in the thin-liquid film two analogous problems on the joint flow of the film and co-current gas flux are considered. In the first problem the generalized condition (7) is used, in the second one we use the classical condition of the heat fluxes equality at \( \Gamma \)

\[
k_y \frac{\partial T_g}{\partial n}(x, h(x)) = k_i \frac{\partial T_i}{\partial n}(x, h(x)).
\]

4. BASIC FLOW

We assume that the Poiseuille’s flow is realized in either layer. The supposition is explained in Remark 1 and allows one to turn to the two-dimensional problem (Pukhnachov, 2006) in the variables \( y \) and \( z \). The exact solution of the problem (1)–(8) has the form

\[
\begin{align*}
\nu_{l,g} &= (0, \nu_{l,g}(z), 0), & p_{l,g} &= p_{l,g}(y, z), & T_{l,g} &= T_{l,g}(y, z).
\end{align*}
\]
In each layer the continuity equation is identically satisfied for the solution (19), other equations are simplified

\[- \frac{1}{\rho} \frac{\partial p_y}{\partial y} = \nu v_{zz}, \quad - \frac{1}{\rho} \frac{\partial p_z}{\partial z} - g = 0, \quad v T_y = \chi (T_{yy} + T_{zz}). \tag{20}\]

Relations (20) are easily integrated, meanwhile the velocities \(v_{l,g}\) are the quadratic functions with respect to the transversal coordinate

\[v(z) = \frac{C_1}{\nu \rho} \frac{z^2}{2} + C_2 z + C_3, \tag{21}\]

the pressure is defined by the formula

\[p(y, z) = -\rho g z + C_1 y + D, \tag{22}\]

the energy equation has a particular solution

\[T(y, z) = B_1 y + B_2 z + B_3 + \frac{B_4}{\chi} \int_{z_0}^{z} (z - \xi) v(\xi) d\xi, \tag{23}\]

For the liquid layer \(z_0 = 0\), then

\[T_l(y, z) = B_{1l} y + B_{2l} z + B_{3l} + \frac{B_{4l} z}{\chi_l} \left( \frac{C_{1l}}{6\nu_l \rho_l} z^3 + \frac{C_{2l}}{2} z^2 + C_{3l} z \right) - \frac{B_{4l}}{\chi_l} \left( \frac{C_{1l}}{8\nu_l \rho_l} z^4 + \frac{C_{2l}}{3} z^3 + \frac{C_{3l}}{2} z^2 \right); \]

for the gas \(z_0 = h(y)\), therefore,

\[T_g(y, z) = B_{1g} y + B_{2g} z + B_{3g} + \frac{B_{4g} z}{\chi_g} \left[ \frac{C_{1g}}{6\nu_g \rho_g} (z^4 - h^3 z) + \frac{C_{2g}}{2} (z^3 - h^2 z) + C_{3g} (z^2 - h z) \right] - \frac{B_{4g}}{\chi_g} \frac{C_{2g}}{8\nu_g \rho_g} (z^4 - h^4) + \frac{C_{3g}}{3} (z^3 - h^3) + \frac{C_{4g}}{2} (z^2 - h^2); \]

Taking into account the kinematic condition

\[\left( \frac{C_{1l}}{\nu_l \rho_l} \frac{h^2(y)}{2} + C_{2l} h(y) + C_{3l} \right) h_y = 0, \tag{24}\]

then \(z = h(y) = h_0\), i.e., the basic flow described by the solution (19) is the stationary flow with a non-deformable interface. It is obvious in this case that the generalized energy condition (14) coincides with the classical one (18). The differences will appear only in the perturbation problem.

Functions \(T_l\) and \(T_g\) define the temperature distributions in liquid and gas phases, respectively. Substitution of these functions into the boundary conditions for the temperature functions (3) on the solid walls leads to the equalities \(B_{1l} = T_{l1}\) and \(B_{1g} = T_{g1}\). Subsequent substitution of \(T_l\) and \(T_g\) into the condition of the temperature continuity at the interface (5) imposes the restriction on values \(T_{l1}\) and \(T_{g1}\); these values must be equal to each other. So, as a consequence of the temperature continuity at \(\Gamma\) the gradients \(T_{l1}\) and \(T_{g1}\) must be equal to each other, \(T_{l1} = T_{g1} = A\).

The order in which the integration constants should be calculated is the following:

\[C_{l2} = \frac{[6\nu_l m_l (h_0^3 - 2\nu h_0^2 - \nu p H^2) + 2\nu h_0^3 A] \nu p}{h_0^2 \nu_l \rho_l \left[ 3(-h_0^3 + 2\nu h_0^3) - 2(2\nu h_0^2 - 2H + h_0) \nu p \right]}; \]

\[C_{l1} = \frac{6\nu_l m_l}{h_0^3} - \frac{3\nu_l \rho_l}{h_0} C_{l2}, \quad C_{l1} = C_{g1}; \]

\[C_{g2} = \frac{-\nu A}{\nu_g \rho_g}; \quad C_{l2} = \frac{C_{l1}}{\nu_g \rho_g} C_{g2}, \quad C_{l3} = 0, \quad C_{g3} = \frac{-C_{g1} H^2}{2\nu_g \rho_g} - C_{g2} H, \]

Interfacial Phenomena and Heat Transfer
of appearing convective motions in the liquid as a result of thermocapillary forces. The primes denoting the non-dimensional variables will further be omitted.

5. STABILITY ANALYSIS

5.1 Form of the Critical Perturbations

To solve the stability problem, non-dimensional variables are introduced. Let \( h_0, h_0^2/\nu_1, v_{1,g}^* = \nu_{1,g}/h_0, \)
\( p_{0}^* = \rho_{l,g} \nu_1^2/h_0^2, T_{l}^* = T_{l0}, \) be the scales of length, time, velocity, pressure, and temperature, respectively. Non-dimensional spatial variables and time have the form \((x', z') = (x, z)/h_0, \tau = \nu_1/t^*,\) respectively. We take the values which are characteristic for the liquid as the measurement units of the media parameters. Thus, the non-dimensional analogue \( \omega_{g,l}^* = \omega_{g,l}/\omega_1 \) is introduced for any parameter of the medium.

According to the specified method of non-dimensionalization the following governing parameters arise:

\[
\begin{align*}
\text{Re}_{l,g} &= \frac{\nu_{l,g}^* h_0}{\nu_1}, & \text{Pr}_{l,g} &= \frac{\nu_{l,g}}{\chi_{l,g}}, & \text{Ga} &= \frac{g h_0^3}{\nu_1^2}, & \text{Ma} &= \frac{\sigma_0 h_0}{\nu_1^2 \rho_{l}}, \\
\text{We} &= \frac{\alpha_0 \nu_1 h_0}{\rho_{l,g} \nu_1^2}, & E &= \frac{\alpha_1 \nu_{l,g}}{k_1 h_0}, & E_2 &= \frac{\alpha_1^2 T_{l0}}{\nu_1 \rho_{l,g} k_1},
\end{align*}
\]

where \( \text{Re} \) and \( \text{Pr} \) are the Reynolds and Prandtl numbers, respectively, \( \text{Ga} \) is the Galileo number defining the ratio between the gravity and viscous forces in the medium, \( \text{Ma} \) is the Marangoni number characterizing the possibility of appearing convective motions in the liquid as a result of thermocapillary mechanism, \( \text{We} \) is the Weber number measuring the ratio of the liquid inertia to the surface tension and characterizing the ability of the interface to deform, \( E, E_2 \) are the energy parameters characterizing the energy which is spent for the interface deformation by the thermocapillary forces. The primes denoting the non-dimensional variables will further be omitted.

Remark 2. The method of non-dimensionalization was chosen taking into account the possibility to compare our theoretical and experimental results given in the form of a map of the flow regimes in the plane of the parameters \((\text{Re}_1, \text{Re}_g)\) (Cheverda and Kabov, 2008).

The small non-stationary perturbations of the pressure \( P(\tau, x, z), \) velocity \( V(\tau, x, z), \) \( W(\tau, x, z), \) temperature \( \theta(\tau, x, z) \) are considered. Linearized in the vicinity of the stationary solution (19) equations (1), (2) for the perturbations in the non-dimensional variables are written in the form

\[
\begin{align*}
U_{l}\tau + v_{l}U_{l} = -P_{l}\tau + \frac{1}{\text{Re}_l} \Delta U_{l}, & \quad V_{l}\tau + v_{l}V_{l} + v_{l}W_{l} = -P_{l}\tau + \frac{1}{\text{Re}_l} \Delta V_{l}, \\
W_{l}\tau + v_{l}W_{l} = -P_{l}\tau + \frac{1}{\text{Re}_l} \Delta W_{l}, & \quad U_{l}\tau + V_{l} + W_{l} = 0, \\
\theta_{\tau} + v_{l}\theta_{l} + V_{l}T_{l}\tau + W_{l}T_{l}\tau = \frac{1}{\text{Pr}_{l,\text{Re}} \text{Re}_l} \Delta \theta;
\end{align*}
\]

are where Lafaurie et al., 1994; Griebel et al., 1998) was used to solve the problem (25)\(\rightarrow\) (30). The initial data for the unknown stability a series of calculations were performed. The modification of the VOF-method (Hirt and Nichols, 1981; longitudinal gradient of the temperature were varied.

The liquid FC-72 and gas nitrogen used in the experiments were chosen as the working media. The values of the critical perturbations is shown in Fig. 3. The patterns presented in Fig. 2(a) and Fig. 3(a) are the thermocapillary

\[ U_{g_x} + v_g U_{g_y} = -P_{g_x} + \frac{1}{Re_g} \Delta U_g, \quad V_{g_x} + v_g V_{g_y} + v_g, W_g = -P_{g_y} + \frac{1}{Re_g} \Delta V_g, \]

\[ W_{g_x} + v_g W_{g_y} = -P_{g_x} + \frac{1}{Re_g} \Delta W_g, \quad U_{g_x} + V_{g_y} + W_{g_x} = 0, \]

\[ \theta_{g_x} + v_g \theta_{g_y} + V_{g_y}T_{g_y} + W_{g_x}T_{g_x} = -\frac{1}{Pr_g Re_g} \Delta \theta_g, \]

with the boundary conditions on the solid walls

\[ U_l(x, 0) = V_l(x, 0) = W_l(x, 0) = 0, \quad \theta_l(x, 0) = 0, \]

\[ U_g(x, H) = V_g(x, H) = W_g(x, H) = 0, \quad \theta_g(x, H) = 0. \]

At the interface the continuity conditions of the velocity and temperature fields and the kinematic condition lead to

\[ U_l = \nu V_g, \quad W_l = \nu W_g, \quad V_l + v_l R = \nu (V_g + v_g R), \]

\[ \theta_l + T_{l_x} R = \theta_g + T_{g_x} R, \quad -v_l R_g + W_l = R_l. \]

After the linearization of the energy condition one has the following equality:

\[ k \frac{\partial^2 \theta}{\partial n^2} \Gamma + k \frac{\partial \theta}{\partial n} \Gamma = k \frac{\partial \theta}{\partial n} \Gamma \cdot n_1 = \alpha T \left[ \nabla \Gamma \cdot \left( \frac{\partial \theta}{\partial n} \Gamma \right) + \nabla \Gamma \cdot \nabla \Gamma \cdot \alpha \left( \nabla \theta + \frac{\partial \theta}{\partial n} \Gamma \right) \right] \cdot \nabla \Gamma \cdot \nu, \]

where \( f \) is the jump of the value of \( f \) on \( \Gamma \),

\[ n_1 = [(R_{Gx} G_{12} - R_{x} G_{22}) e_1 + (R_{x} G_{12} - R_{e} G_{11}) e_2] / (G_{11} G_{22} - G_{12}^2). \]

Then, the energy balance condition in the non-dimensional form is written as

\[ k T_s (\theta_{g_x} + T_{g_x} R) - \theta_{l_x} - T_{l_x} R = (E + E_2 T) / \nu (U_{g_x} + V_{g_y} - U_{l_x} - V_{l_y} + W_{l}). \]

Here, \( T_s = T_{g_x} / T_{l_x} \). The normal and tangential components of the dynamic condition in the linear approximation are

\[ -\nu^2 P_{g} + P_l + 2(\nu^2 W_{g_x} - W_{l_x}) - (\nu^2 v_{g_x} - v_{l_x}) R_y = (\nu^2 v_{g} - v_{l}) R + \left[ \text{We} - \text{Ma} T \right] (R_{xx} + R_{yy}), \]

\[ \nu^2 (U_{g_x} + W_{g_y}) - (U_{l_x} + W_{l_y}) = -\text{Ma} (\theta_{x} - T_l R_x), \]

\[ \nu^2 (V_{g_x} + W_{g_y}) - (V_{l_x} + W_{l_y}) = -\text{Ma} (\theta_{y} - T_l R_y). \]

To determine the character and degree of the influence of the variations in the internal energy on the film flow stability a series of calculations were performed. The modification of the VOF-method (Hirt and Nichols, 1981; Lafaurie et al., 1994; Griebel et al., 1998) was used to solve the problem (25)\(\rightarrow\) (30). The initial data for the unknown functions were 1\%–3\% from the averaged values of the corresponding characteristics of the basic flow.

The liquid FC-72 and gas nitrogen used in the experiments were chosen as the working media. The values of the physical parameters are presented in Table 1. In the computation the mass flow rate of the liquid and values of the longitudinal gradient of the temperature were varied.

It was established that the structure of the critical disturbances at the interface liquid – gas changed under the external effects of different intensity (Fig. 2). The structure of the velocity and temperature fields for all types of the critical perturbations is shown in Fig. 3. The patterns presented in Fig. 2(a) and Fig. 3(a) are the thermocapillary
TABLE 1: Values of the physical parameters for FC-72 – nitrogen system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>FC-72</th>
<th>Nitrogen</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρ, kg/m³</td>
<td>1.68 × 10³</td>
<td>1.2</td>
</tr>
<tr>
<td>ν, m²/s</td>
<td>0.4 × 10⁻⁶</td>
<td>0.15 × 10⁻⁴</td>
</tr>
<tr>
<td>β, K⁻¹</td>
<td>1.4 × 10⁻³</td>
<td>3.67 × 10⁻³</td>
</tr>
<tr>
<td>k, W/(m×K)</td>
<td>0.48</td>
<td>0.02717</td>
</tr>
<tr>
<td>χ, m²/s</td>
<td>0.3 × 10⁻⁷</td>
<td>0.3 × 10⁻⁴</td>
</tr>
<tr>
<td>α, N/(m×K)</td>
<td>11.03 × 10⁻⁵</td>
<td></td>
</tr>
<tr>
<td>σ₀, N/m</td>
<td>8.6 × 10⁻³</td>
<td></td>
</tr>
</tbody>
</table>

longitudinal rolls (LR) generated by the liquid motion along the surface tension gradients. In Fig. 2(b) and Fig. 3(b) the transversal rolls (TR) are shown. TRs are presumably induced by hydrodynamic modes and observed under sufficiently large gas flow rates. These rolls drift in the direction opposite to the basic flow. The interface deformation amplitude and transversal size of TR increase with the increasing gas flow rate. At large gas flow rates the deviations of the interface from the undisturbed form can reach 10% of the thickness h₀. In Fig. 2(c) the hybrid structures in the form of “spots” are presented. These patterns arise due to the interaction of the thermal and hydrodynamic perturbations. Typical velocity and temperature distributions in the planes (x, z) (in cross section y = 35) and (y, z) (in cross section x = 25) are presented in Fig. 3(c) and Fig. 3(d), respectively.

The problem (25)–(30) with the zero right-hand side of condition (29) [it corresponds to the classical condition of the energy balance at the interface (18) was also solved (see Figs. 4 and 5; results were obtained under the same values of parameters as Figs. 2 and 3)]. Form of the critical perturbations are not changed principally (Fig. 4).

In this cases width of LRs are more than in the first problem and patterns of the thermal and hydrodynamical perturbations leading to formation of TRs are distinct from structures in the problem with the generalized condition (29). Unlike the first problem, here the appearing TRs are immovable (standing) and driven by the main flux along the flow axis at rather large gas flow rates. With increasing Re, the transversal size of TRs and the interface deformation amplitude increase.

5.2 Instability Regimes

To define the critical characteristics of the stability and conditions of structure generation in the form of longitudinal, transversal waves and hybrid “spots,” one assumes that the perturbations of the exact solution (19) are the normal spatial ones

\[ (V, P, T) = (V_\ast(z), P_\ast(z), T_\ast(z)) e^{i(-Ct+α_1 x+α_2 y)}, ~ α_1, α_2 ∈ \mathbb{R}. \]  

(31)

The substitution of (31) into (25)–(30) leads to the characteristic boundary problem. A nontrivial solution of this problem exists given the defined values of the spectral parameter C, which is the eigenvalue of the problem and describes the behavior of the disturbance with time. In their turn, the corresponding eigenfunctions determine the structure of the characteristic perturbations of the velocity, temperature, and pressure. In the general case, the disturbances (31) are the harmonic oscillations given by the amplitude and phase. Here, the absolute values of the real and imaginary parts of the functions V_\ast(x), P_\ast(x), T_\ast(x) are the amplitudes of the cosinusoidal and sinusoidal components, with the first one specifying the perturbation amplitude and the second one defining the phase. Let M be a set of the known parameters of the problem, then, the eigenvalues and eigenfunctions depend on M and wave numbers α₁, α₂ along the axes x and y, respectively. In the general case, C are the complex numbers of the form C = C_r + IC_i, and the perturbations propagate in the flux in the form of waves at an angle ϕ = α₁/α₂ in the direction of y with the common wave number α = √α₁² + α₂² and phase velocity ω = C_r/α. Damping or growing of the perturbations is determined by the sign of the imaginary part C_i; the value C_r prescribes the oscillation frequency. If the decrement C is a purely
FIG. 2: Critical perturbations: (a) — longitudinal waves, $Re_g = 50$, $Re_l = 3.51$, $Ma = 8.27 \times 10^{-3}$; (b) — transversal waves, $Re_g = 200$, $Re_l = 4.2$, $Ma = 19.71 \times 10^{-3}$; (c) — coexistence of longitudinal and transversal waves, $Re_g = 100$, $Re_l = 20.7$, $Ma = 37.92 \times 10^{-3}$.

FIG. 3: Typical distributions of the velocity and temperature fields in the film: (a) — $Re_g = 50$, $Re_l = 3.51$, $Ma = 8.27 \times 10^{-3}$; (b) — $Re_g = 200$, $Re_l = 4.2$, $Ma = 19.71 \times 10^{-3}$; (c), (d) — $Re_g = 100$, $Re_l = 20.7$, $Ma = 37.92 \times 10^{-3}$.

imaginary number, then with $C_i < 0$ the perturbations will monotonically damp, with $C_i > 0$ they will grow. The conditions $C_i(M, \alpha) = 0$, $C_r(M, \alpha) = 0$ define the stability boundaries with respect to the monotonic and oscillatory perturbations, correspondingly.

To calculate the neutral curve and threshold characteristics of the stability of the basic flow in the problems with the full problem 1 and classical energy condition problem 2 (the right hand side of (29) is equal to zero) the Abramov – Godunov method (Godunov, 1961; Abramov, 1962) adapted for the considered problems according to Andreev et al. (2000); Gaponov and Maslov (1980) was used. The typical structure of the neutral curves in the plane (Ma, $\alpha_j$) is presented in Fig. 6. The curves separating domains $I$ in both problems correspond to the monotonic instability of the “spiral” perturbations with the zero wave number along the component $y$. In this case the thermocapillary instability is observed, which is revealed by the formation of the longitudinal rolls [Fig. 3(a), scheme of this regime is presented in Fig. 7(a)]. The increasing wave number leads to the splitting of the neutral curve into monotonic and oscillatory branches [Fig. 6(a)]. The oscillatory instability is peculiar for the plane perturbations with $U = 0$. The differences
Influence of Internal Energy Variations of the Interface

FIG. 4: Critical perturbations in the problem with the classical energy condition: (a) — longitudinal waves, \( \Re_g = 50, \Re_l = 3.51, \Ma = 8.27 \times 10^{-3} \); (b) — transversal waves, \( \Re_g = 200, \Re_l = 4.2, \Ma = 19.71 \times 10^{-3} \); (c) — coexistence of longitudinal and transversal waves, \( \Re_g = 100, \Re_l = 20.7, \Ma = 37.92 \times 10^{-3} \)

FIG. 5: Typical distributions of the velocity and temperature fields in the film in the problem with the classical energy condition: (a) — \( \Re_g = 50, \Re_l = 3.51, \Ma = 8.27 \times 10^{-3} \); (b) — \( \Re_g = 200, \Re_l = 4.2, \Ma = 19.71 \times 10^{-3} \); (c), (d) — \( \Re_g = 100, \Re_l = 20.7, \Ma = 37.92 \times 10^{-3} \)

become apparent in the types of the instability in zones II (problem 1) and III (problem 2). In domain II the crisis of the flow is generated by the oscillatory mode (running wave) and accompanied by the appearance of the rolls running across the main flow (running transversal rolls, RTR) moving in the direction opposite to the flow axes [Fig. 3(b) and scheme in Fig. 7(b)].

In problem 2 quite another transition from the thermocapillary instability to another regime is observed. In zone III the instability is revealed as the standing transversal rolls (STR) [Fig. 5(b) and scheme in Fig. 7(c)]. The neutral curve has an additional branch of the monotonic instability [Fig. 6(b)]. In this case the critical perturbation is a standing wave. It is evident that the standing wave is a particular case of the running wave traveling at a zero phase velocity. The main difference from the running one is that the energy transfer does not occur in the standing wave. It is only the transition from one kind of energy into another which occurs. Whereas the running wave when propagating in the medium transfers the energy.

The energy spent on the interface deformation by the thermocapillary forces is characterized by the values \( E \sim O(10^{-6}), E_2 \sim O(10^{-3}) \). Taking into account this effect results in changing the global minimum of the neutral
curve and most dangerous modes, with the parameter $S = 18.57$ for the considered media. In problem 1 the local minimum point is $(M_a^*, \alpha_1^*, \alpha_2^*) = (1.78 \times 10^{-3}, 0, 6.77)$, and the instability is generated by the shortwave plane perturbations. In problem 2 this point is $(M_a^*, \alpha_1^*, \alpha_2^*) = (0.63 \times 10^{-2}, 0, 0)$, and the flow is absolutely unstable with respect to the thermocapillary “spiral” perturbations.

### 5.3 Comparison with the Experiments

The obtained results were compared with the experimental data. The flows of the thin liquid film driven by the gas flux in the microchannel of the height $H = 1.4$ mm with the initial thickness of the film 50–100 µm were experimentally studied (Cheverda and Kabov, 2008). The experimental map of the regimes arising in the system FC-72 – nitrogen is presented in Fig. 8(a) in the plane of the parameters $Re_g$, $Re_l$. The theoretical map of the instability regimes is shown in Fig. 8(b).

A curve running from $(Re_g, Re_l) = (100, 5)$ to $(1200, 18)$ in Fig. 8(a) separates domain in the lower part of the map, where a film rupture was observed in experiments. The rupture was usually similar to a long narrow strip along film flow. It could change $Re_l$ in the domain wetted by a liquid and influence on the surface curvature across the flow. Therefore, experiments were not performed with values of $Re_g$, $Re_l$ which are lower than the curve.

The aim of the investigation was not to predict a film rupture. For this, the van der Waals forces and contact line dynamics must be taken into account in a mathematical model. The model and analysis of the model are much more complex and difficult. Consequently, the suggested approach in our manuscript cannot predict the curve running from...
Influence of Internal Energy Variations of the Interface

FIG. 8: Comparison of the maps of the flow regimes: (a) — experimental map, (b) — theoretical map for problem 1 at $Ma = 8.27 \times 10^{-3}$, $We = 3199.4$

\((Re_g, Re_l) = (100, 5) \text{ to } (1200, 18)\) in Fig. 8(a). Therefore, results of calculations obtained in the framework of our approach and experimental data did not compare with respect to the parameter. For more details concerning the film rupture we refer to Ajaev (2013), Ajaev and Kabov (2017).

According to the results of the experiments small-scale perturbations of the interface are observed in zone $A$ [cells in Fig. 9(A)]. On the theoretical map the domain $A$ corresponds to the region of the stability. But, we note that in Fig. 8(b) the characteristic “cape” is at small $Re_l$. In the zone $B$ the proposed treatment allows one to predict the formation of the longitudinal rolls at the qualitative level [Fig. 3(a) and Fig. 9(B)]. We also forecast the coexistence of the longitudinal and transversal rolls [Figs. 9(C) and 10(C)] moving against the main flow in the zone and the appearance of the hybrid “spots” [Fig. 2(c)]. In the experiments the coexistence of the longitudinal and transversal irregular structures [Figs. 9(D) and 10(D)] is observed in the region $D$. In the framework of the linear theory the analytical description of this regime is impossible.

The quantitative differences are observed since the infinite channel is used in the mathematical model, whereas the experiments are carried out with the working segment of the finite length. Furthermore, the working medium FC-72 is evaporated during the experiment, and the mass transfer processes at the interface are not taken into account in the mathematical model used. Finally, in the experiments the substrate is kept at constant temperature, but in the theoretical consideration the investigated exact solution takes into account the fact that the longitudinal temperature gradients are applied on the outer channel walls. The application of the Dirichlet conditions for the temperature function in the study of unidirectional flow directly imposes the restriction that the temperature function in the exact solution must be constant. Nevertheless, the obtained qualitative agreement of the results is the reason for the performed theoretical investigation.

FIG. 9: Experiment, snapshots of the regimes arising in the zones $A$ (cells), $B$ (longitudinal structures), $C$ (2D waves), $D$ (3D waves)
6. CONCLUSION

The stability of the flow of a thin liquid film driven by a gas flux has been studied in the framework of the model with the generalized energy condition at the interface. The most dangerous mode has been specified and a map of flow regimes has been constructed. A comparison with the experimental data has been performed. In the generalized energy condition the deficit of heat, transferred through the interface, has been taken into account. The deficit characterizes the energy consumption for the interface deformation. Taking into account this additional effect results in the qualitative agreement of the obtained theoretical results with the known experimental results for the media FC-72 (liquid) and nitrogen (gas).

The goal of the manuscript was to explain the nature and mechanisms of spatial structure formation. We assume only qualitative coincidence concerning the mechanisms of instability and appearing regimes of the instability. According to our theoretical results the instabilities can appear in the form of the thermocapillary (longitudinal) rolls and coexistence of the thermocapillary (longitudinal) and moving transversal rolls. And besides, the transversal rolls move in the direction opposite to the gas flow direction. In the experiments precisely these regimes were also observed. Classical theory predicts a regime of coexistence of the thermocapillary (longitudinal) and standing transversal rolls. The regime was not observed in the experiments. We think that such agreement between obtained theoretical and experimental results is quite convincing to regard the proposed approach as adequate one and describing the principal features which cannot be found with help of the classical theory.

The formation of structures of any types in the film is caused by the action of the thermocapillary forces. The interaction of the hydrodynamic and thermal perturbations with the basic flow always leads to the generation of thermocapillary longitudinal rolls. At rather large gas flow rates the transversal rolls additionally appear in the film. The invariance of the problem for perturbations with regard to the change of the wave number \( \alpha_1 \) for \( -\alpha_1 \) means that the perturbations leading to the formation of the thermocapillary rolls propagate along \( x \) in the opposite directions with the equal phase velocities. It means that in the case of the problem symmetry with respect to every couple \((\pm \alpha_1, \alpha_2)\) for the considered flow one observes the formation of the stationary longitudinal perturbations (thermocapillary rolls). For any couple \((\alpha_1, \alpha_2)\) and \((\alpha_1, -\alpha_2)\) the symmetry of the corresponding spectral problem is broken, therefore, the arising transversal rolls fitting the perturbations with the wave numbers \( \pm \alpha_2 \) interact with the basic flow in a different way. The perturbations propagating downstream (correspond to \( \alpha_2 \)) are merged by the basic flow. The perturbations with the wave number \( -\alpha_2 \) generate the transversal rolls traveling against the flux.

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Influence of Internal Energy Variations of the Interface


Goncharova, O. N., Hennenberg, M., Rezanova, E. V., and Kabov, O. A., Modeling of the convective fluid flows with evaporation


APPENDIX

Presenting formulation of the thermal condition on a free boundary, we follow the derivation of the conditions on the interface between liquids in Andreev et al., 2012; Chapter 2.

Let us introduce some notations. Let a fluid fill a domain $\Omega \subset \mathbb{R}^3$. We assume that $\omega_t \subset \Omega$ is an arbitrary fluid volume having a non-empty intersection $\gamma_t$ with the interface $\Gamma_t$, $\partial \omega_t$ is its boundary, $\omega_{1t}$ and $\omega_{2t}$ are the parts of $\omega_t$ that belong to $\Omega_1$ and $\Omega_2$, respectively (Fig. A1). Let us denote $\Sigma_{it} = \partial \omega_{it} / \gamma_t$ ($i = 1, 2$), $n_i$ is the unit vector of the outward (with regard to $\omega_t$) normal to the surface $\Sigma_{it}$. The line $\partial \gamma_t$ is the boundary of domain $\gamma_t$; $n_\Gamma$ is the unit vector defined at each point of $\partial \gamma_t$; this vector lies in the tangential plane to the surface $\gamma_t$ at this point and, simultaneously, in the normal plane to the curve $\partial \gamma_t$ (the direction $n_\Gamma$ is external relative to the domain $\omega_t$). The surface $\gamma_t$ and the curve $\partial \gamma_t$ are assumed to have continuous curvature; $\partial \omega_t$ can be a piecewise-smooth surface.

Energy of a material volume of a fluid is the sum of its kinetic and internal energy. The volumetric density of the kinetic energy is $\rho | \mathbf{v} |^2 / 2$, the volumetric density of the internal energy is $\rho U_{en}$, where $U_{en}$ is the additive function of sets, which is called the internal energy. Total energy of the moving volume $\Omega$ is

$$E_t(\Omega) = \int_{\Omega} \rho \left( \frac{| \mathbf{v} |^2}{2} + U_{en} \right) d\Omega.$$
Volume energy change can ensue both due to work of the external and internal forces and due to action of heat sources. Accordingly, the law conservation of the total energy has the following form:

$$\frac{d}{dt} E(\omega_t) = \frac{d}{dt} \int_{\omega_t} \rho\left(\frac{|\mathbf{v}|^2}{2} + U_{en}\right) d\omega + \int_{\partial\omega_t} \mathbf{v} \cdot \mathbf{p}_n d\Sigma + \int_{\partial\omega_t} q_n d\Sigma + \int_{\omega_t} \rho h d\omega, \quad (A1)$$

where $\mathbf{f}(\mathbf{x}, t)$, $\mathbf{p}_n(\mathbf{x}, t)$ are the densities of mass and surface forces, respectively, $\mathbf{x} = (x, y, z)$, $q_n(\mathbf{x}, t)$ is the surface density of the heat flux incoming into the volume through $\partial\omega_t$, $h(\mathbf{x}, t)$ is the volume density of internal (e.g., radioactive or chemical) heat sources.

Let us postulate the integral law of conservation of the total energy of a system of immiscible fluids enclosed in a moving volume $\omega_t$ in the form

$$\frac{d}{dt} \int_{\omega_t} \rho_i\left(\frac{|\mathbf{v}_i|^2}{2} + U_{en}\right) d\omega_i + \frac{d}{dt} \int_{\omega_{2t}} \rho_2\left(\frac{|\mathbf{v}_2|^2}{2} + U_{en}\right) d\omega_2 + \frac{d}{dt} \int_{\gamma_t} U_u d\gamma = \int_{\omega_{1t}} \rho_1 \mathbf{v}_1 \cdot \mathbf{f}_1 d\omega_1 + \int_{\omega_{2t}} \rho_2 \mathbf{v}_2 \cdot \mathbf{f}_2 d\omega_2 + \int_{\Sigma_1} \mathbf{v}_1 \cdot \mathbf{P}_1 \cdot \mathbf{n}_1 d\Sigma_1 + \int_{\Sigma_2} \mathbf{v}_2 \cdot \mathbf{P}_2 \cdot \mathbf{n}_2 d\Sigma_2 + \int_{\gamma_t} \mathbf{q}_1 \cdot \mathbf{n}_1 d\gamma_1 - \int_{\gamma_t} \mathbf{q}_2 \cdot \mathbf{n}_2 d\gamma_2$$

$$+ \int_{\omega_{1t}} \rho_1 h_1 d\omega_1 + \int_{\omega_{2t}} \rho_2 h_2 d\omega_2. \quad (A2)$$

Here $U_u$ is the specific surface internal energy, $\mathbf{v}$ is the common limiting value of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ on the surface $\Gamma$, $\mathbf{P}_i$ is the stress tensor in the $i$th fluid, $dl$ is a linear element of the curve $\partial\gamma_t$, $\sigma$ is the coefficient surface tension [in our case $\sigma$ is temperature-dependent scalar function, see (4)], $\mathbf{q}_i$ is the heat-flux vector in the $i$th fluid. The additional [as compared with (A1)] term in the left-hand side of equality (A2) expresses the rate of energy of the selected part of the interface, whereas the curvilinear integral in the right-hand side of this relation is the power of capillary forces spent on deformation of the mentioned surface.

Equality (A2) admits considerable simplifications. Let us note that the equations of continuity, momentum, and heat inflow are satisfied in each subdomain $\omega_{it}$ where the functions $\rho_i$, $\mathbf{v}_i$, $T_i$ are sufficiently smooth. Therefore, the identities

$$\frac{d}{dt} \int_{\omega_{1t}} \rho_i\left(\frac{|\mathbf{v}_i|^2}{2} + U_{en}\right) d\omega_i - \int_{\omega_{1t}} \rho_i \mathbf{v}_i \cdot \mathbf{f}_i d\omega_i - \int_{\omega_{1t}} \rho_i h_1 d\omega_i$$

$$- \int_{\Sigma_1} \mathbf{v}_i \cdot \mathbf{P}_i \cdot \mathbf{n}_i - \mathbf{q}_i \cdot \mathbf{n}_i d\Sigma_1 = (-1)^{i-1} \int_{\gamma_t} \mathbf{v}_i \cdot \mathbf{P}_i \cdot \mathbf{n}_i - \mathbf{q}_i \cdot \mathbf{n}_i d\gamma_1 \quad (A3)$$

are valid for $i = 1, 2$. In writing the right-hand side of (A3), we used the notation $\mathbf{v}_i|_\Gamma = \mathbf{v}$ based on the continuity condition [see the first condition in (5)]. Taking into account (A2) and (A3), we obtain

$$\frac{d}{dt} \int_{\gamma_t} U_u d\gamma = \int_{\gamma_t} (\mathbf{q} \cdot \mathbf{n} - \mathbf{v} \cdot (\mathbf{P} \cdot \mathbf{n})) d\gamma + \int_{\partial\gamma_t} \mathbf{qv} \cdot \mathbf{n}_\Gamma d\ell. \quad (A4)$$

The symbol $[f]$ is used hereinafter to indicate the difference between the traces $f_1|_\Gamma$ and $f_2|_\Gamma$ of the functions $f_1$ and $f_2$ determined in the domains $\Omega_1$ and $\Omega_2$, respectively, $[f] = f_1 - f_2$.

Further transformations are aimed at presenting the left-hand side of (A4) and the last term in the right-hand side of this equality in the form of certain integrals over the domain $\gamma_t$. The transformation of the first expression is based
on an analog of the translation formula. Let \( \gamma_t \) be a smooth material surface, and let \( f(x, t) \) be a smooth function determined in a certain neighborhood of this surface. Then, we obtain

\[
\frac{d}{dt} \int_{\gamma_t} f \, d\gamma = \int_{\gamma_t} \left( \frac{df}{dt} + f \text{div}_v \right) \, d\gamma, \tag{A5}
\]

where \( \frac{df}{dt} \) is the total derivative of the function \( f \), and \( \text{div}_v \) is the surface divergence of the vector \( v \), which was determined in Section 2.

Similar to the classical translation formula

\[
\frac{d}{dt} \int_{\omega_t} f \, d\omega = \int_{\omega_t} \left( \frac{df}{dt} + f \text{div} \right) \, d\omega,
\]

relation (A5) is proved by passing to the Lagrange coordinates. Let the surface \( \gamma_0 \) at \( t = 0 \) be defined by the parametrization \( x = x_0 (\xi^1, \xi^2) \), where \( \xi^1, \xi^2 \) are curvilinear coordinates, and the coordinate \( \xi^3 \) is counted along the normal to the surface \( \gamma_0 \). The triple of the numbers \( \xi^1, \xi^2, \xi^3 \) determines the Lagrange coordinates, at least, in a certain neighborhood of \( \gamma_0 \). As \( \gamma_t \) is a moving surface, its location at a current time instant \( t \) is determined as a constriction of the mapping \( \xi \rightarrow x \) formed by solving the Cauchy problem

\[
\frac{dx}{dt} = v(x, t), \quad x|_{t=0} = \xi.
\]

Let us pass to the Lagrange coordinates in the left-hand side of (A5). Then, the integral whose derivative has to be calculated takes the form

\[
\int_{\gamma_0} \tilde{f}(\xi(t), t) \tilde{J}_\Gamma(\xi(t), t) \, d\gamma_0, \tag{A6}
\]

where \( \tilde{f}(\xi(t), t) = f(x(\xi(t), t), t) \), and \( \tilde{J}_\Gamma \) is the determinant of the metric tensor of the surface \( \gamma_0 \), and the following equality is valid:

\[
\tilde{J}_\Gamma(\xi(t), t) = \sqrt{\det g} \, |\xi(t)|^{1/2}.
\]

The term on the right-hand side of the last equality \( g = |e_1 \times e_2| \) is the determinant of the metric tensor of the surface \( \Gamma \) with the elements \( g_{\alpha\beta} = e_\alpha \cdot e_\beta \) (\( \alpha, \beta = 1, 2 \)). The vectors

\[
e_\alpha = \frac{\partial x}{\partial \xi^\alpha} \quad (\alpha = 1, 2)
\]

are covariant vectors of the tangential basis to this surface. For a fixed value of \( \xi \), function \( \tilde{J}_\Gamma \) satisfies the differential equation (according to the Euler formula)

\[
\frac{d\tilde{J}_\Gamma}{dt} = \tilde{J}_\Gamma \text{div}_v \nabla_v.
\]

Let us prove the validity of (A7). As \( \tilde{J}_\Gamma = (\det g_{\alpha\beta})^{1/2} \), then

\[
\frac{d\tilde{J}_\Gamma}{dt} = \frac{1}{2\sqrt{\det g}} \frac{\partial g}{\partial x^{\alpha}} \frac{d g_{\alpha\beta}}{dt} = \frac{1}{2\sqrt{\det g}} \frac{g^{\beta\alpha}}{g_{\alpha\beta}} \frac{d g_{\alpha\beta}}{dt}, \tag{A8}
\]

where \( g^{\beta\alpha} \) are the elements of a matrix inverse to \( g_{\alpha\beta} \). Further, as \( g_{\alpha\beta} = (\partial x / \partial \xi^\alpha) \cdot (\partial x / \partial \xi^\beta) \), then

\[
\frac{d g_{\alpha\beta}}{dt} = \frac{\partial v}{\partial \xi^\alpha} \cdot e_\beta + e_\alpha \cdot \frac{\partial v}{\partial \xi^\beta} = (e_\alpha \cdot \nabla_v v) \cdot e_\beta + (e_\beta \cdot \nabla_v v) \cdot e_\alpha, \tag{A9}
\]
where \( e_\alpha \) are the covariant vectors of the tangential basis of the surface \( \Gamma \), and \( \nabla_\Gamma \) is the surface gradient determined by the following formula:

\[
\nabla_\Gamma = e^\alpha \frac{\partial}{\partial \xi^\alpha}.
\]

(A10)

The symbol \( \nabla_\Gamma v \) denotes a tensor of the form (summation is performed over the superscript \( \beta \))

\[
\nabla_\Gamma v = e^\beta \otimes \frac{\partial v}{\partial \xi^\beta},
\]

where \( \otimes \) denotes a dyadic product. Let \( D_\Gamma \) be the symmetric part of the tensor: \( 2D_\Gamma = \nabla_\Gamma v + (\nabla_\Gamma v)^* \). Then equality (A9) is rewritten in the form

\[
\frac{dg_{\alpha\beta}}{dt} = 2e^\beta \cdot D_\Gamma \cdot e_\alpha.
\]

From the last relation and (A8), we obtain

\[
\frac{dJ_\Gamma}{dt} = \nabla_\Gamma e_\alpha \cdot D_\Gamma \cdot e_\alpha.
\]

To derive (A7) we have to prove that

\[
e^\alpha \cdot D_\Gamma \cdot e_\alpha = \text{div}_\Gamma v.
\]

(A11)

Let us denote the normal component of the vector \( v \) by \( v^3 \) (let us recall that the directions of the \( \xi^3 \) axis and the normal \( n \) coincide); then we obtain \( v = v^\gamma e_\gamma + v^3 n = v_\Gamma + v^3 n \). We determine the tensor \( D'_\Gamma \) through the relation

\[
2D'_\Gamma = \nabla_\Gamma (v^3 n) + (\nabla_\Gamma (v^3 n))^*.
\]

Thus, we have an obvious chain of equalities:

\[
e^\alpha \cdot D'_\Gamma \cdot e_\alpha = \frac{1}{2} e^\alpha \cdot e^\beta \otimes \left( \frac{\partial v^3}{\partial \xi^\beta} n + v^3 \frac{\partial n}{\partial \xi^\beta} \right) \cdot e_\alpha + \frac{1}{2} e^\alpha \cdot \left( \frac{\partial v^3}{\partial \xi^\beta} n + v^3 \frac{\partial n}{\partial \xi^\beta} \right) \otimes e^\beta \cdot e_\alpha = \frac{1}{2} e^\alpha \cdot e^\beta \otimes \left( \frac{v^3}{\partial \xi^\beta} \right) \cdot e_\alpha + \frac{1}{2} e^\alpha \cdot \left( \frac{v^3}{\partial \xi^\beta} \right) = v^3 \text{div}_\Gamma n + n \cdot \nabla_\Gamma v^3 = \text{div}_\Gamma (v^3 n).
\]

In a similar manner, we find that the following relation is valid for the tensor \( D''_\Gamma = D_\Gamma - D'_\Gamma \):

\[
e^\alpha \cdot D''_\Gamma \cdot e_\alpha = \text{div}_\Gamma v_\Gamma.
\]

This and previous equalities show that formula (A11) and, correspondingly, (A7) are valid. Based on (A7), we can obtain formula (A5).

The proof of (A5) is based on passing to the Lagrange coordinates in the integral in the left-hand side by formula \( x = x(\xi, t) \). The derivative of the integrand in (A6) with respect to \( t \), by virtue of the Euler formula analogue (A7) and the definition of the total derivative is

\[
\frac{\partial}{\partial t} (f \circ J_\Gamma) = \frac{d}{dt} (f J_\Gamma) = \left( \frac{df}{dt} + f \text{div}_\Gamma v \right) J_\Gamma.
\]

The reverse transition from the Lagrange to the Euler variables yields (A5).

To transform the curvilinear integral in the right-hand side of (A4), we use the formula

\[
\int_{\partial \gamma_t} b \cdot n_\Gamma \, dl = \int_{\gamma_t} \text{div}_\Gamma [b - (n \cdot b)n] \, d\gamma_t,
\]

(A12)
where \( b \) is an arbitrary smooth vector. To prove the validity of (A12), we note that the following relation holds by virtue of the identity \( \mathbf{n} \times \mathbf{d}l = -\mathbf{n} \times \mathbf{d}l \):

\[
\int_{\partial \gamma_t} b \cdot \mathbf{n} \mathbf{d}l = \int_{\partial \gamma_t} (\mathbf{n} \times b) \cdot \mathbf{d}l,
\]

(A13)

where \( \mathbf{d}l \) is the oriented element of the tangent line to the curve \( \partial \gamma_t \). We transform the integral in the right-hand side of (A13) on the basis of the Stokes theorem by taking into account the equality

\[
\mathbf{n} \cdot [\text{rot}(\mathbf{n} \times b)] = \text{div}[\mathbf{b} - (\mathbf{n} \cdot \mathbf{b})\mathbf{n}],
\]

(A14)

In turn, equality (A14) follows from the vector analysis formula

\[
\text{rot}(\mathbf{n} \times b) = \mathbf{b} \cdot \nabla \mathbf{n} - \mathbf{n} \cdot \nabla \mathbf{b} + \mathbf{n} \text{ div} \mathbf{b} - \mathbf{b} \text{ div} \mathbf{n}
\]

and from the fact that \( |\nabla \mathbf{n}|^2 = 0 \), and, therefore, \( \mathbf{n} \cdot (\mathbf{b} \cdot \nabla \mathbf{n}) = 0 \). Finally, we have

\[
\text{div} [\mathbf{b} - (\mathbf{n} \cdot \mathbf{b})\mathbf{n}] = \text{div}_\Gamma [\mathbf{b} - (\mathbf{n} \cdot \mathbf{b})\mathbf{n}],
\]

because the vector \( \mathbf{b} - (\mathbf{n} \cdot \mathbf{b})\mathbf{n} \) has a zero normal component. The required equality (A12) follows from here and from (A13), (A14).

Let us now assume that \( f = U_u \) in (A5) and \( \mathbf{b} = \sigma \mathbf{v} \) in (A12). Using these formulas, we write both sides of equality (A4) as integrals over the domain \( \gamma_t \). We obtain

\[
\int_{\gamma_t} (\frac{dU_u}{dt} + U_u \text{div}_\Gamma \mathbf{v}) \, d\gamma = \int_{\gamma_t} (\{\mathbf{q} \cdot \mathbf{n}\} - \mathbf{v} \cdot [\mathbf{P} \cdot \mathbf{n}] + \text{div}_\Gamma (\sigma \{\mathbf{v} - (\mathbf{n} \cdot \mathbf{v})\mathbf{n}\})) \, d\gamma.
\]

(A15)

As the domain \( \gamma_t \subset \Gamma \) is arbitrary, it follows from (A15) that the integrands in both sides of this equality coincide. In other words,

\[
\frac{dU_u}{dt} + U_u \text{div}_\Gamma \mathbf{v} = \{\mathbf{q} \cdot \mathbf{n}\} - \mathbf{v} \cdot [\mathbf{P} \cdot \mathbf{n}] + \text{div}_\Gamma (\sigma \mathbf{v}_\Gamma),
\]

(A16)

where the notation \( \mathbf{v}_\Gamma = \mathbf{v} - (\mathbf{n} \cdot \mathbf{v})\mathbf{n} \) is used.

To find the final form of condition (A16), we use the following relations: equality

\[
U_u(T) = \sigma - T \frac{d\sigma}{dT};
\]

(A17)

Fourier law \( \mathbf{q} = -k \nabla T \); dynamic condition on the interface (8), which implies that

\[
\mathbf{v} \cdot [\mathbf{P} \cdot \mathbf{n}] = 2\sigma K (\mathbf{n} \cdot \mathbf{v}) + \mathbf{v}_\Gamma \cdot \nabla \sigma;
\]

formula

\[
\text{div}_\Gamma (\sigma \mathbf{v}_\Gamma) = \sigma \text{div}_\Gamma \mathbf{v}_\Gamma + \mathbf{v}_\Gamma \cdot \nabla \sigma;
\]

equality

\[
\text{div}_\Gamma \{\mathbf{n} \cdot \mathbf{v}\mathbf{n}\} = -2K (\mathbf{n} \cdot \mathbf{v}),
\]

which follows from relation \( \text{div}_\Gamma \mathbf{n} = -2K \). Based on all considerations discussed above, we can conclude that

\[
-T \sigma''(T) \frac{dT}{dt} = T \sigma'(T) \text{div}_\Gamma \mathbf{v} - \left[ k \frac{\partial T}{\partial n} \right],
\]

(A18)

where the prime denote differentiation with respect to \( T \).
**Remark 3.** Equality (A17) follows from the Gibbs—Duhem equation $d\sigma + s\,dT = 0$ and relation $\sigma = U_u - T\,s$, where $s$ is the entropy.

Condition (A18) can be called the energy condition on the interface $\Gamma_t$ of two fluids. It means that the jump of the heat flux into direction of the normal to $\Gamma_t$ is compensated by the change in the internal energy of this surface. In turn, this change is related both to the change in temperature (and, correspondingly, in the specific internal energy) and to the change in the interface area: this circumstance is responsible for the emergence of the first term in the right-hand side of (A18). In an important particular case of a linear dependence of $\sigma$ on $T$ of the form (4), condition (A18) is simplified to

$$[k\frac{\partial T}{\partial n}] + \alpha T \text{div}_I \mathbf{v} = 0. \quad (A19)$$

Finally, if $\sigma = \text{const}$, then (A18) expresses the continuity of the surface density of the heat flux $q_n = k\partial T/\partial n$ across the interface.