DIFFERENTIAL CONSTRAINTS FOR THE PROBABILITY DENSITY FUNCTION OF STOCHASTIC SOLUTIONS TO THE WAVE EQUATION

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By using functional integral methods we determine new types of differential constraints satisfied by the joint probability density function of stochastic solutions to the wave equation subject to uncertain boundary and initial conditions. These differential constraints involve unusual limit partial differential operators and, in general, they can be grouped into two main classes: the first one depends on the specific field equation under consideration (i.e., on the stochastic wave equation), the second class includes a set of intrinsic relations determined by the structure of the joint probability density function of the wave and its derivatives. Preliminary results we have obtained for stochastic dynamical systems and first-order nonlinear stochastic particle differential equations (PDEs) suggest that the set of differential constraints is complete and, therefore, it allows determining uniquely the probability density function of the solution to the stochastic problem. The proposed new approach can be extended to arbitrary nonlinear stochastic PDEs and it could be an effective way to overcome the curse of dimensionality for random boundary and initial conditions. An application of the theory developed is presented and discussed for a simple random wave in one spatial dimension.

KEY WORDS: *stochastic partial differential equations, high-dimensional methods, random fields*

1. INTRODUCTION

Many physical phenomena such as sound propagation, electromagnetic scattering at random surfaces, and random vibrations in solid mechanics can be described in terms of random waves satisfying the standard wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = U^2 \nabla^2 \psi \tag{1}$$

for random boundary conditions, random initial conditions, or random group velocity. The purpose of this paper is to introduce a new method to compute the statistical properties of the stochastic solutions to Eq. (1). This method is based on a set of differential constraints for the joint probability density function of the wave ψ and its derivatives and it may present an advantage with respect to more classical stochastic approaches such as polynomial chaos [1–4], probabilistic collocation [5, 6], and generalized spectral decompositions [7–11]. In fact, it seems that it does not suffer from the curse of dimensionality problem, at least when randomness comes from boundary or initial conditions. Indeed, since we are solving for probability density function of the system, we can actually prescribe these conditions in terms of the probability distributions and this is obviously not dependent on the number of random variables characterizing the underlying probability space.

This observation immediately leads us to the question: Is it possible to determine a closed evolution equation for the probability density function of the solution to Eq. (1) at a specific space-time location? Unfortunately, the answer is negative. In fact, the self-interacting nature of the wave equation is associated with nonlocal solutions in space and time that, in turn, yield the impossibility of determining a pointwise equation for the probability density function.

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However, an equation for the probability density functional of the wave always can be obtained. This very general approach relies on the use of functional integral techniques [12–16]; in particular, those involving the Hopf characteristic functional (see Appendix B). These functional methods aim to cope with the global probabilistic structure of the solution and they have been extensively studied in the past as a possible tool to tackle many fundamental problems in physics such as turbulence [17, 18]. Their usage grew very rapidly in the 1970s, when it became clear that diagrammatic functional techniques [12] could be applied, at least formally, to many different problems in classical statistical physics. However, functional differential equations involving the Hopf characteristic functional unfortunately are not amenable to numerical simulation. In addition, the amount of statistical information carried on by the Hopf characteristic functional is often far beyond the needs of practical uncertainty quantification, which usually reduces only to the computation of a few statistical moments of the solution at specific space-time locations.

Thus, we are led to look for alternative ways to determine the evolution of the probability density function associated with the solution to Eq. (1). To this end, by using a functional integral technique we have introduced recently in the context of stochastic dynamical systems, in this paper we will show that it is possible to formulate a set of differential constraints that have to be satisfied locally by the probability density function of every random wave process governed by Eq. (1). These constraints involve, in general, unusual limit partial differential operators and they can be grouped into two main classes: the first one depends on the specific field equation under consideration; i.e., on the stochastic wave equation; the second class is represented by a set of intrinsic relations arising from the structure of the joint probability density function of the wave and its derivatives. A fundamental question we address in this paper is whether the set of these differential constraints allows us to determine uniquely the probability density function associated with the stochastic solution to the wave equation.

This paper is organized as follows. In Section 2 we consider random waves in one spatial dimension and we show how to determine a closed and exact differential constraint that has to be satisfied by the probability density function of every random wave process for random boundary conditions and random initial conditions. All the calculations are based on a functional integral technique that is presented in detail in Appendix A. Section 3 deals with the formulation of additional differential constraints depending on the structure of the joint probability density function of the wave and its derivatives. The completeness of the set of differential constraints is discussed in Section 4 for a prototype problem involving a first-order nonlinear stochastic particle differential equation (PDE). In Section 5 we generalize the theory developed for two- and three-dimensional random wave processes. An example of the application of the theory is presented in Section 6. Finally, the main findings and their implications are summarized in Section 7. We also include two additional appendices. The first one (i.e., Appendix B), deals with the application of the Hopf characteristic functional approach to one-dimensional random waves. The second one (i.e., Appendix C), deals with the perturbation expansion of differential constraints in the neighborhood of a selected space-time location.

2. ONE-DIMENSIONAL RANDOM WAVES

In one spatial dimension Eq. (1) can be written as

$$\frac{\partial^2 \psi}{\partial t^2} = U^2 \frac{\partial^2 \psi}{\partial x^2} \,. \tag{2}$$

We supplement this equation with appropriate random boundary conditions and with a random initial condition in a suitable space-time domain. The solution to this boundary value problem is clearly a random wave, which we assume to admit a probability density function. At this point a fundamental question is whether we can actually determine an evolution equation for such a probability density based on Eq. (2). To this end, let us first look for a representation of the joint probability of the wave and its first-order derivatives with respect to space and time calculated at different space-time locations. The reason why we need such joint probability density will become clear in a while. For notational convenience, let us set

$$\psi \overset{\text{def}}{=} \psi(x,t;\omega)\,, \qquad \psi'_t \overset{\text{def}}{=} \frac{\partial \psi}{\partial t}(x',t';\omega)\,, \qquad \psi''_x \overset{\text{def}}{=} \frac{\partial \psi}{\partial x}(x'',t'';\omega)\,. \tag{3}$$

This allows us to write the joint probability density of ψ, ψ'_t, and ψ''_x as (see Appendix A for further details)

$$p^{(a,b,c)}_{\psi\psi'_t\psi''_x} \stackrel{\text{def}}{=} \langle \underbrace{\delta\left(a-\psi\right)}_{\text{constructor field}} \underbrace{\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)}_{\text{absorbing fields}} \rangle\,, \tag{4}$$

where δ denotes the Dirac delta function and the average operator $\langle\cdot\rangle$ is defined as a functional integral with respect to the joint measure of the random initial condition and the random boundary conditions. The definition we give of "constructor field" relies on the fact that we will extract from that Dirac delta function, also called the "indicator function" by Klyatskin [19, p. 42]; all the derivatives we need to build up the wave equation. As we will see, this procedure also generates additional terms that can be treated in a limit process by employing the "absorbing fields". By using the differentiation rules for the probability density function discussed in Appendix A we obtain

$$\frac{\partial p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial t} = -\frac{\partial}{\partial a}\langle\delta\left(a-\psi\right)\psi_t\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle \tag{5}$$

and

$$\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial t^2} = \frac{\partial^2}{\partial a^2}\langle\delta\left(a-\psi\right)\psi_t^2\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle - \frac{\partial}{\partial a}\langle\delta\left(a-\psi\right)\psi_{tt}\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle\,. \tag{6}$$

Similarly,

$$\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial x^2} = \frac{\partial^2}{\partial a^2}\langle\delta\left(a-\psi\right)\psi_x^2\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle - \frac{\partial}{\partial a}\langle\delta\left(a-\psi\right)\psi_{xx}\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle\,. \tag{7}$$

If we subtract Eq. (7) from Eq. (6) and we take Eq. (2) into account we obtain

$$\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial t^2} - U^2\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial x^2} = \frac{\partial^2}{\partial a^2}\langle\delta\left(a-\psi\right)\left(\psi_t^2-U^2\psi_x^2\right)\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle\,. \tag{8}$$

The average appearing in Eq. (8) can be easily represented if we take the limit for (x', x'') and (t', t'') going to x and t, respectively. In fact, by using the absorbing fields appearing in Eq. (4) we see that

$$\left(b^2-U^2c^2\right)p^{(a,b,c)}_{\psi\psi_t\psi_x} = \lim_{\substack{t',t''\to t\\ x',x''\to x}}\langle\delta\left(a-\psi\right)\left(\psi_t^2-U^2\psi_x^2\right)\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle\,. \tag{9}$$

This yields the final result

$$\lim_{\substack{t',t''\to t\\ x',x''\to x}}\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial t^2} = U^2\lim_{\substack{t',t''\to t\\ x',x''\to x}}\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial x^2} + \left(b^2-U^2c^2\right)\frac{\partial^2 p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a^2}\,. \tag{10}$$

Note that this identity involves unusual partial differential operators that we shall call limit partial derivatives. For subsequent mathematical developments it is convenient to reserve a special symbol for these operators; i.e., we shall define

$$\frac{\mathfrak{d}^2 p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}t^2} \stackrel{\text{def}}{=} \lim_{\substack{t',t''\to t\\ x',x''\to x}}\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial t^2}\,, \qquad \frac{\mathfrak{d}^2 p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}x^2} \stackrel{\text{def}}{=} \lim_{\substack{t',t''\to t\\ x',x''\to x}}\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial x^2}\,. \tag{11}$$

This allows us to write Eq. (10) in a compact form as

$$\frac{\mathfrak{d}^2 p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}t^2} = U^2\frac{\mathfrak{d}^2 p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}x^2} + \left(b^2-U^2c^2\right)\frac{\partial^2 p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a^2}\,. \tag{12}$$

Equation (12) is a differential constraint that must be satisfied by the probability density function of every one-dimensional random wave governed by Eq. (2) for the random initial condition or random boundary conditions. These types of differential constraints made their first appearance in [20] in the context of stochastic dynamical systems. Note that Eq. (12) always involves only three variables (x, t, a) and two parameters (b and c) independently of the dimensionality of the random boundary conditions and the random initial condition. This very attractive feature immediately leads us to the question: Is it possible to solve a boundary value problem for Eq. (12) and, therefore, obtain the joint probability density function of the random wave and its derivatives at a specific space-time location uniquely? Certainly, if we would have this possibility then we would be able to overcome the curse of dimensionality problem. However, as we will see in Section 6, there exist an infinite number of probability densities satisfying the differential constraint (12) for the same boundary and initial conditions. This suggests that a boundary value problem for Eq. (12) is, in general, ill posed.

A possible way to get rid of the limit partial derivatives is to integrate Eq. (12) with respect to the variables b and c. This procedure yields the following equation for the response probability, i.e., for the probability density of the wave ψ:

$$\frac{\partial^2 p_{\psi}^{(a)}}{\partial t^2} = U^2 \frac{\partial^2 p_{\psi}^{(a)}}{\partial x^2} + \frac{\partial^2}{\partial a^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(b^2 - U^2 c^2\right) p_{\psi \psi_t \psi_x}^{(a,b,c)} \, db dc \,, \tag{13}$$

where

$$p_{\psi(x,t)}^{(a)} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{\psi \psi_t \psi_x}^{(a,b,c)} \, db dc \tag{14}$$

denotes the marginal density of $p_{\psi \psi_t \psi_x}^{(a,b,c)}$. The integrals appearing in Eq. (14) are formally written from $-\infty$ to ∞ although the probability density function we integrate out may be compactly supported. Note that Eq. (13) is not closed. In fact, the last term on the right-hand side can be written as

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(b^2 - U^2 c^2\right) p_{\psi \psi_t \psi_x}^{(a,b,c)} \, db dc = \left\langle \delta(a-\psi) \int_{-\infty}^{\infty} b^2 \delta(b-\psi_t) db \int_{-\infty}^{\infty} \delta(c-\psi_x) dc \right\rangle \\ &-U^2 \left\langle \delta(a-\psi) \int_{-\infty}^{\infty} \delta(b-\psi_t) db \int_{-\infty}^{\infty} c^2 \delta(c-\psi_x) dc \right\rangle = \left\langle \delta(a-\psi)\psi_t^2 \right\rangle - U^2 \langle \delta(a-\psi)\psi_x^2 \rangle \,. \end{aligned} \tag{15}$$

The last two averages of this equation involve the calculation of the correlations between two functionals of the random initial conditions and the random boundary conditions; e.g., $\delta(a-\psi)$ and ψ_t^2 . Such a correlation structure can be disentangled, for example, by using a functional power series [21, 22] (see also [23], p. 311, and [19], p. 70). In other words, the correlations $\langle \delta(a-\psi)\psi_t^2 \rangle$ and $\langle \delta(a-\psi)\psi_x^2 \rangle$ can be represented in terms of cumulants of the random initial conditions and the random boundary conditions. Alternatively, other types of closures can be considered [24–27].

We conclude this section by emphasizing that Eq. (13) can be obtained more directly from the representation of the response probability of the wave; i.e.,

$$p_{\psi(x,t)}^{(a)} = \langle \delta[a - \psi(x,t)] \rangle \,. \tag{16}$$

In fact, if we differentiate Eq. (16) twice with respect to t and x

$$\frac{\partial^2 p_{\psi(x,t)}^{(a)}}{\partial t^2} = -\frac{\partial}{\partial a} \langle \delta(a-\psi)\psi_{tt} \rangle + \frac{\partial^2}{\partial a^2} \langle \delta(a-\psi)\psi_t^2 \rangle \,, \tag{17}$$

$$\frac{\partial^2 p_{\psi(x,t)}^{(a)}}{\partial x^2} = -\frac{\partial}{\partial a} \langle \delta(a-\psi)\psi_{xx} \rangle + \frac{\partial^2}{\partial a^2} \langle \delta(a-\psi)\psi_x^2 \rangle \tag{18}$$

and sum up the derivatives, we immediately obtain

$$\frac{\partial^2 p_{\psi(x,t)}^{(a)}}{\partial t^2} = U^2 \frac{\partial^2 p_{\psi(x,t)}^{(a)}}{\partial x^2} + \frac{\partial^2}{\partial a^2} \langle \delta(a-\psi)\psi_t^2 \rangle - U^2 \frac{\partial^2}{\partial a^2} \langle \delta(a-\psi)\psi_x^2 \rangle \,. \tag{19}$$

This shows that the differential constraint (12) includes the evolution equation of the response probability of the wave as a subcase.

2.1 Some Remarks on Evolution Equations Involving Limit Partial Differential Operators

Equation (12) involves unusual limit partial differential operators in space and time. These operators arise very often when the functional integral method or, equivalently, the Hopf characteristic functional approach (see Appendix B) are applied to a stochastic PDE in order to determine a pointwise equation for the probability density function of its solution. In this section we would like to provide some heuristic justification of why this happens. To this end, let us first rewrite the one-dimensional wave Eq. (2) as a first-order system

$$\frac{\partial\psi}{\partial t} = U^2\frac{\partial\eta}{\partial x}, \qquad \frac{\partial\eta}{\partial t} = \frac{\partial\psi}{\partial x}. \tag{20}$$

This formulation clearly shows that the temporal evolution of the wave ψ is driven by the random field $\partial\eta/\partial x$ while the temporal evolution of η is governed by $\partial\psi/\partial x$. In other words, the stochastic dynamics of random waves is, in general, self-interacting. A similar situation arises in the simpler context of time-dependent stochastic dynamical systems of order greater than 1, where the dynamics of the stochastic solution is influenced by the dynamics of higher-order time derivatives. A self-interacting system can yield to nonlocal solutions in space and time. Perhaps, the simplest example in this context is the diffusion equation. In this case, a pointwise equation for the probability density function cannot be derived in a closed form. However, there are a few exceptions where we can actually obtain a closed equation for the probability density function. Among them, we recall first-order nonlinear and quasilinear stochastic PDEs. In these cases the limit partial derivatives do not appear and the evolution problem for the probability density function is well defined in a classical sense. A prime example is the inviscid Burgers equation

$$\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi}{\partial x} = 0 \tag{21}$$

with random boundary or initial conditions. In this case it can be shown that the probability density function of the solution satisfies

$$\frac{\partial p^{(a)}_{\psi(x,t)}}{\partial t} + a\frac{\partial p^{(a)}_{\psi(x,t)}}{\partial x} + \int_{-\infty}^{a}\frac{\partial p^{(a')}_{\psi(x,t)}}{\partial x}da' = 0\,. \tag{22}$$

3. INTRINSIC RELATIONS DEPENDING ON THE STRUCTURE OF THE JOINT PROBABILITY DENSITY FUNCTION

The fields appearing in joint probability density (4) are related locally to each other and, therefore, we expect that there exists a certain number of relations between the limit partial derivatives of the joint probability density with respect to different arguments. These relations are independent of the particular stochastic field equation describing the physical phenomenon [e.g., Eq. (2)], but they are defined intrinsically by the structure of the joint probability density function. For instance, there is clearly a local deterministic relation between the random field ψ and its spatial derivative

$$\psi_x(x,t;\omega) \overset{\text{def}}{=} \lim_{x'\to x}\frac{\psi(x',t;\omega)-\psi(x,t;\omega)}{x'-x}\,. \tag{23}$$

This relation, and similar ones for other derivatives, can be translated into an intrinsic relation involving joint probability density function (4). To this end, we first observe that the limit derivatives of Eq. (4) with respect to different arguments (e.g., x, x', and x'') are different. Indeed,

$$\frac{\mathfrak{d} p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d} x} = -\frac{\partial}{\partial a}\left\langle\delta\left(a-\psi\right)\psi_x\delta\left(b-\psi_t\right)\left(c-\psi_x\right)\right\rangle = -c\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a}\,, \tag{24}$$

$$\frac{\mathfrak{d} p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d} x'} = -\frac{\partial}{\partial b}\left\langle\delta\left(a-\psi\right)\delta\left(b-\psi_t\right)\psi_{tx}\delta\left(c-\psi_x\right)\right\rangle, \tag{25}$$

$$\frac{\mathfrak{d} p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d} x''} = -\frac{\partial}{\partial c}\left\langle\delta\left(a-\psi\right)\delta\left(b-\psi_t\right)\delta\left(c-\psi_x\right)\psi_{xx}\right\rangle. \tag{26}$$

Similarly,

$$\frac{\mathfrak{d}p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}t} = -\frac{\partial}{\partial a}\langle \delta\,(a-\psi)\,\psi_t\delta\,(b-\psi_t)\,(c-\psi_x)\rangle = -b\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a}\,, \tag{27}$$

$$\frac{\mathfrak{d}p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}t'} = -\frac{\partial}{\partial b}\langle \delta\,(a-\psi)\,\delta\,(b-\psi_t)\,\psi_{tt}\delta\,(c-\psi_x)\rangle\,, \tag{28}$$

$$\frac{\mathfrak{d}p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}t''} = -\frac{\partial}{\partial c}\langle \delta\,(a-\psi)\,\delta\,(b-\psi_t)\,\delta\,(c-\psi_x)\,\psi_{xt}\rangle\,. \tag{29}$$

From Eqs. (25) and (29) it follows that

$$\int_{-\infty}^{b}\frac{\mathfrak{d}p^{(a,b',c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}x'}db' = \int_{-\infty}^{c}\frac{\mathfrak{d}p^{(a,b,c')}_{\psi\psi_t\psi_x}}{\mathfrak{d}t''}dc'\,. \tag{30}$$

Next, let us consider the average of the second-order spatial derivative of ψ. A comparison between Eq. (26) and Eq. (7) (in the limit for $x', x'' \to x$ and $t', t'' \to t$) immediately gives

$$\int_{-\infty}^{a}\frac{\mathfrak{d}^2p^{(a',b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}x^2}da' = c^2\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a} + \int_{-\infty}^{c}\frac{\mathfrak{d}p^{(a,b,c')}_{\psi\psi_t\psi_x}}{\mathfrak{d}x''}dc'\,. \tag{31}$$

Similarly, a comparison between Eqs. (28) and (6) yields

$$\int_{-\infty}^{a}\frac{\mathfrak{d}^2p^{(a',b,c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}t^2}da' = b^2\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a} + \int_{-\infty}^{b}\frac{\mathfrak{d}p^{(a,b',c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}t'}db'\,. \tag{32}$$

Equations (24), (27), and (30)–(32) are simple examples of intrinsic relations involving the probability density function. These relations are in the form of differential constraints and they have to be satisfied independently of the particular stochastic problem under consideration; e.g., Eq. (2). In principle, if the wave field ψ is analytic, by repeatedly applying the arguments above we can construct an infinite set of differential constraints to be satisfied at every space-time location by the joint probability density function of the wave and its derivatives. Note that from Eqs. (28), (26), and (2) we easily obtain that

$$\int_{-\infty}^{b}\frac{\mathfrak{d}p^{(a,b',c)}_{\psi\psi_t\psi_x}}{\mathfrak{d}t'}db' = U^2\int_{-\infty}^{c}\frac{\mathfrak{d}p^{(a,b,c')}_{\psi\psi_t\psi_x}}{\mathfrak{d}x''}dc'\,, \tag{33}$$

which is another, more compact, way to write the differential constraint for the probability density function associated with wave Eq. (2). Indeed, a substitution of Eqs. (31) and (32) into Eq. (33) consistently gives Eq. (12).

4. ON THE COMPLETENESS OF THE SET OF DIFFERENTIAL CONSTRAINTS

In some cases, it can be proven that the set of differential constraints involving the first-order limit partial derivatives is equivalent to a standard evolution equation for the joint probability density function of the system. This circumstance implies that the set of differential constraints is complete; i.e., that it allows for the computation of the joint density. This happens, for example, in the context of high-order stochastic dynamical systems subject to a random initial state. The set of differential constraints, in this case, is equivalent to the well-known Liouville equation. In this section we would like to prove a similar result for first-order nonlinear stochastic PDEs. To this end, let us consider the equation

$$\frac{\partial\psi}{\partial t} + \left(\frac{\partial\psi}{\partial x}\right)^2 = 0 \tag{34}$$

subject to random boundary and initial conditions of arbitrary dimensionality. The solution to this problem can be represented in probability space by using the joint density

$$p_{\psi\psi'_x}^{(a,b)} = \langle \delta(a-\psi)\delta(b-\psi'_x)\rangle \, . \tag{35}$$

The full set of first-order differential constraints associated with Eqs. (34) and (35) is easily obtained as

$$\frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} t} = b^2 \frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial a} \, , \tag{36}$$

$$\frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} t'} = \frac{\partial}{\partial b}\left(2b\langle \delta(a-\psi)\delta(b-\psi_x)\psi_{xx}\rangle\right) \, , \tag{37}$$

$$\frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} x} = -b \frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial a} \, , \tag{38}$$

$$\frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} x'} = -\frac{\partial}{\partial b}\langle \delta(a-\psi)\delta(b-\psi_x)\psi_{xx}\rangle \, . \tag{39}$$

From Eqs. (37) and (39) it follows that

$$\frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} t'} = -\frac{\partial}{\partial b}\left(2b\int_{-\infty}^{b} \frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b')}}{\mathfrak{d} x'} db'\right) \, . \tag{40}$$

At this point, let us recall the following identities between standard partial derivatives and limit partial derivatives

$$\frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial t} = \frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} t} + \frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} t'} \, , \tag{41}$$

$$\frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial x} = \frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} x} + \frac{\mathfrak{d} p_{\psi\psi_x}^{(a,b)}}{\mathfrak{d} x'} \, . \tag{42}$$

Note that at the left hand side we first set $x = x'$ and $t = t'$ and then we differentiate while at the right hand side we first differentiate and then we set $x = x'$ and $t = t'$. By substituting Eqs. (36) and (40) into Eq. (41) and using Eqs. (42) and (38), we obtain

$$\frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial t} = b^2 \frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial a} - \frac{\partial}{\partial b}\left[2b\left(\int_{-\infty}^{b} \frac{\partial p_{\psi\psi_x}^{(a,b')}}{\partial x} db' + \int_{-\infty}^{b} b' \frac{\partial p_{\psi\psi_x}^{(a,b')}}{\partial a} db'\right)\right] \, . \tag{43}$$

This is the correct evolution equation for the joint probability density function associated with the stochastic solution to Eq. (34). We can show this by considering Eq. (35), with both ψ and ψ_x set at the same space-time location. A differentiation with respect to time yields, in this case,

$$\frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial t} = b^2 \frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial a} + \frac{\partial}{\partial b}\left(2b\langle \delta(a-\psi)\delta(b-\psi_x)\psi_{xx}\rangle\right) \, , \tag{44}$$

where the last term can be written in terms of the probability density function by inverting the relation

$$\frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial x} = -b \frac{\partial p_{\psi\psi_x}^{(a,b)}}{\partial a} - \frac{\partial}{\partial b}\langle \delta(a-\psi)\delta(b-\psi_x)\psi_{xx}\rangle \, , \tag{45}$$

i.e.,

$$\langle \delta(a-\psi)\delta(b-\psi_x)\psi_{xx}\rangle = -\int_{-\infty}^{b} \frac{\partial p_{\psi\psi_x}^{(a,b')}}{\partial x} db' - \int_{-\infty}^{b} b' \frac{\partial p_{\psi\psi_x}^{(a,b')}}{\partial a} db' \,. \tag{46}$$

A substitution of Eq. (46) into Eq. (44) gives exactly Eq. (43). Thus, we have shown that the set of first-order differential constraints (36)–(39) is equivalent to a standard PDE for joint probability density function (35) on the hyperplane $x' = x$, $t' = t$. This implies that the set of differential constraints, in this case, is complete. However, it is still an open question if such a set is complete for more general stochastic problems; e.g., for the stochastic wave Eq. (2). The dynamics of the joint probability density function in these cases does not satisfy a standard PDE. This is due to the fact that such dynamics, in general, arises from the projection of a functional differential equation on the hyperplane $x = x' = x'' = \cdots$, $t = t' = t'' = \cdots$. In some sense, the set of differential constraints we have obtained represents the components of such a projection.

5. TWO- AND THREE-DIMENSIONAL RANDOM WAVES

The theoretical apparatus developed in the previous sections can be easily extended to two- and three-dimensional wave equations. To this end, let us first consider two spatial dimensions and the equation

$$\frac{\partial^2\psi}{\partial t^2} = U^2\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) . \tag{47}$$

The joint probability density function of the random wave and its derivatives has the functional integral representation

$$p_{\psi\psi'_t\psi''_x\psi'''_y}^{(a,b,c,d)} = \langle \underbrace{\delta\left(a-\psi\right)}_{\text{constructor field}} \underbrace{\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\delta\left(c-\psi'''_y\right)}_{\text{absorbing fields}}\rangle \,. \tag{48}$$

As before, we now differentiate the constructor field with respect to time and space variables and then we use the absorbing fields in a limit procedure in order to get rid of the additional terms arising from the differentiation process. We obtain,

$$\begin{aligned}\frac{\partial^2 p_{\psi\psi'_t\psi''_x\psi'''_y}^{(a,b,c,d)}}{\partial t^2} &= \frac{\partial^2}{\partial a^2}\langle \delta\left(a-\psi\right)\psi_t^2\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\delta\left(d-\psi'''_y\right)\rangle \\ &-\frac{\partial}{\partial a}\langle \delta\left(a-\psi\right)\psi_{tt}\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\delta\left(d-\psi'''_y\right)\rangle \,,\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\partial^2 p_{\psi\psi'_t\psi''_x\psi'''_y}^{(a,b,c,d)}}{\partial x^2} &= \frac{\partial^2}{\partial a^2}\langle \delta\left(a-\psi\right)\psi_x^2\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\delta\left(d-\psi'''_y\right)\rangle \\ &-\frac{\partial}{\partial a}\langle \delta\left(a-\psi\right)\psi_{xx}\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\delta\left(d-\psi'''_y\right)\rangle \,,\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial^2 p_{\psi\psi'_t\psi''_x\psi'''_y}^{(a,b,c,d)}}{\partial y^2} &= \frac{\partial^2}{\partial a^2}\langle \delta\left(a-\psi\right)\psi_y^2\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\delta\left(d-\psi'''_y\right)\rangle \\ &-\frac{\partial}{\partial a}\langle \delta\left(a-\psi\right)\psi_{yy}\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\delta\left(d-\psi'''_y\right)\rangle \,.\end{aligned} \tag{51}$$

A summation of Eqs. (49)-(51) and a following limit procedure for (x', x'', x''') and (t', t'', t''') going to x and t, respectively, yields

$$\frac{\partial^2 p_{\psi\psi_t\psi_x\psi_y}^{(a,b,c,d)}}{\partial t^2} = U^2\left(\frac{\partial^2 p_{\psi\psi_t\psi_x\psi_y}^{(a,b,c,d)}}{\partial x^2} + \frac{\partial^2 p_{\psi\psi_t\psi_x\psi_y}^{(a,b,c,d)}}{\partial y^2}\right) + \left[b^2 - U^2\left(c^2+d^2\right)\right]\frac{\partial^2 p_{\psi\psi_t\psi_x\psi_y}^{(a,b,c,d)}}{\partial a^2} \,. \tag{52}$$

that is the differential constraint for the joint probability density function of every two-dimensional random wave satisfying wave Eq. (47) for random boundary or random initial conditions. Straightforward extensions of the arguments above yield the following differential constraint for the joint probability density function of three-dimensional random waves:

$$\frac{\partial^2 p^{(a,b,c,d,e)}_{\psi\psi_t\psi_x\psi_y\psi_z}}{\partial t^2} = U^2 \left(\frac{\partial^2 p^{(a,b,c,d,e)}_{\psi\psi_t\psi_x\psi_y\psi_z}}{\partial x^2} + \frac{\partial^2 p^{(a,b,c,d,e)}_{\psi\psi_t\psi_x\psi_y\psi_z}}{\partial y^2} + \frac{\partial^2 p^{(a,b,c,d,e)}_{\psi\psi_t\psi_x\psi_y\psi_z}}{\partial z^2} \right) + \left[b^2 - U^2\left(c^2+d^2+e^2\right)\right] \frac{\partial^2 p^{(a,b,c,d)}_{\psi\psi_t\psi_x\psi_y}}{\partial a^2} . \tag{53}$$

Differential constraints (52) and (53) can be supplemented with additional intrinsic relations involving the joint probability density function. These relations can be constructed by following the same mathematical steps presented in Section 3.

6. AN EXAMPLE

Let us consider a random wave in one spatial dimension satisfying the following boundary value problem:

$$\begin{cases} \dfrac{\partial^2\psi}{\partial t^2} = U^2 \dfrac{\partial^2 \psi}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geq t_0 \\ \psi(x,t_0;\omega) = \displaystyle\sum_{k=1}^{3} \xi_k(\omega) h_k(x) \\ \psi_t(x,t_0;\omega) = 0 \end{cases} \tag{54}$$

where $\xi_k(\omega)$ ($k = 1, ..., 3$) are random variables with known joint probability density functions and $h_k(x)$ are prescribed deterministic functions on the real line. The analytical solution to problem (54) is the well-known d'Alambert wave [28, p. 41]

$$\psi(x,t;\omega) = \frac{1}{2} \sum_{k=1}^{3} \xi_k(\omega) \left[h_k\left(x+Ut\right) + h_k\left(x-Ut\right)\right] . \tag{55}$$

For subsequent mathematical developments, it is convenient to define

$$G_{1k} \overset{\text{def}}{=} \frac{1}{2} \left[h_k\left(x+Ut\right) + h_k\left(x-Ut\right)\right] , \tag{56}$$

$$G'_{2k} \overset{\text{def}}{=} \frac{\partial G_{1k}}{\partial t}(x',t') , \tag{57}$$

$$G''_{3k} \overset{\text{def}}{=} \frac{\partial G_{1k}}{\partial x}(x'',t'') . \tag{58}$$

This allows us to write solution (55) and its derivatives with respect to x and t in a compact form as

$$\psi = \sum_{k=1}^{3} \xi_k G_{1k} , \quad \psi'_t = \sum_{k=1}^{3} \xi_k G'_{2k} , \quad \psi''_x = \sum_{k=1}^{3} \xi_k G''_{3k} . \tag{59}$$

The joint probability density of ψ, ψ'_t, and ψ''_x (i.e., the joint probability of the wave and its first-order derivatives at different space-time locations) can be obtained by using the classical mapping approach. To this end, let us notice that Eqs. (59) define a three-dimensional linear transformation from $\{\xi_1, \xi_2, \xi_3\}$ to $\{\psi, \psi'_x, \psi''_t\}$. In matrix-vector form this transformation can be represented as

$$\begin{pmatrix} \psi \\ \psi'_t \\ \psi''_x \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G'_{21} & G'_{22} & G'_{23} \\ G''_{31} & G''_{32} & G''_{33} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} . \tag{60}$$

The determinant of the system's matrix is

$$J \stackrel{\text{def}}{=} G_{11}\left(G'_{22}G''_{33} - G''_{32}G'_{23}\right) + G_{12}\left(G''_{31}G'_{23} - G'_{21}G''_{33}\right) + G_{13}\left(G'_{21}G''_{32} - G''_{31}G'_{22}\right) , \tag{61}$$

while the transpose of its algebraic complement has entries

$$\begin{aligned} &H_{11} = G'_{22}G''_{33} - G''_{32}G'_{23} , && H_{12} = G_{13}G''_{32} - G_{12}G''_{33} , && H_{13} = G_{12}G'_{23} - G'_{22}G_{13} , \\ &H_{21} = G''_{31}G'_{23} - G'_{21}G''_{33} , && H_{22} = G_{11}G''_{33} - G_{13}G''_{31} , && H_{23} = G_{13}G'_{21} - G_{11}G'_{23} , \\ &H_{31} = G'_{21}G''_{32} - G''_{31}G'_{22} , && H_{32} = G''_{31}G_{12} - G_{11}G''_{32} , && H_{33} = G_{11}G'_{22} - G'_{21}G_{12} . \end{aligned} \tag{62}$$

A classical result of probability theory [29, p. 183] states that if $p^{(\alpha_1,\alpha_2,\alpha_3)}_{\xi_1\xi_2\xi_3}$ is the joint probability density of ξ_1, ξ_2, and ξ_3, then the joint probability of ψ, ψ'_t, and ψ''_x is given by

$$p^{(a,b,c)}_{\psi\psi'_x\psi''_t} = \frac{1}{|J|} p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3} , \tag{63}$$

where

$$\widehat{\alpha}_i(a,b,c,x,x',x'',t,t',t'') = \frac{1}{J}\left(H_{i1}a + H_{i2}b + H_{i3}c\right) . \tag{64}$$

Note that every $\widehat{\alpha}_i$ is also a function of all the space-time variables because of the matrix components H_{ij}. Therefore, Eq. (63) is a rather complex function of nine variables. Under the assumption that $\xi_i(\omega)$ are mutually independent normal random variables, we can apply the convolution theorem and obtain the probability density function of the random wave ψ, as

$$p^{(a)}_{\psi(x,t)} = \frac{1}{\sqrt{2\pi}\,|\sigma(x,t)|} e^{-a^2/[2\sigma(x,t)^2]} , \qquad \sigma(x,t)^2 = G_{11}^2 + G_{12}^2 + G_{13}^2 , \tag{65}$$

where G_{1i} are given in Eq. (56). This probability density is shown in Fig. 1 at different time instants for $U = 2$ and an initial condition defined in terms of the functions

$$h_1(x) = \frac{3}{2}e^{-x^2/2} , \qquad h_2(x) = \sin(x^2) , \qquad h_3(x) = \cos(3x) . \tag{66}$$

According to Eq. (65) the mean wave is zero, while the variance $\sigma(x,t)^2$ evolves in time as shown in Fig. 2.

6.1 Analytical Verification of Some Differential Constraints

It is easy to prove that probability density function (63) verifies simple differential constraints (24) and (27); hereafter, rewritten for convenience as

$$\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial x} = -c\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a} , \qquad \frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial t} = -b\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a} .$$

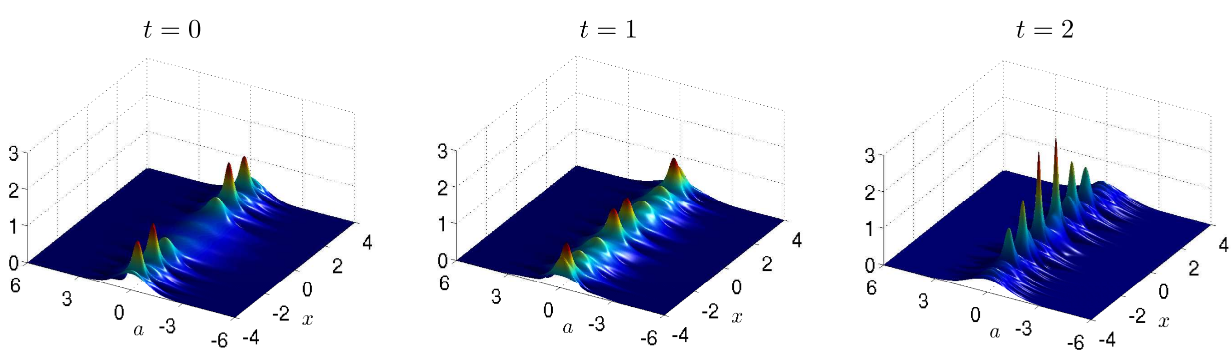

FIG. 1: Probability density function $p^{(a)}_{\psi(x,t)}$ of the random wave at different time instants.

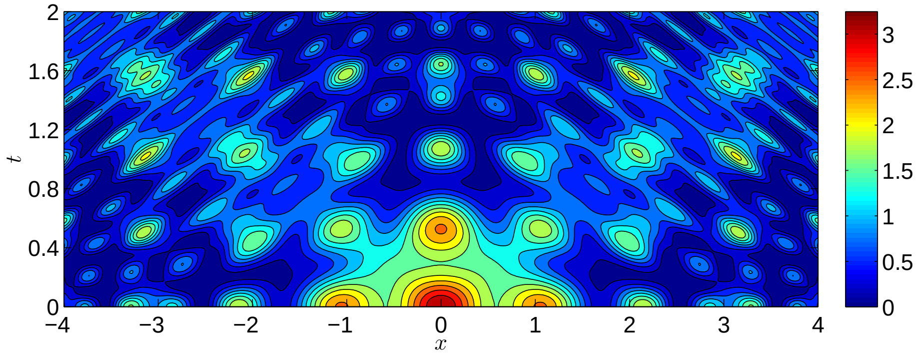

FIG. 2: Temporal evolution of the wave variance.

To this end, let us first notice that the limit partial derivatives of Jacobian determinant (61) with respect to x and t are identically zero; i.e., we have

$$\frac{\partial J}{\partial x} = 0\,, \qquad \frac{\partial J}{\partial t} = 0\,. \tag{67}$$

In fact, by using definitions (57) and (58) we obtain

$$\lim_{\substack{t',t''\to t\\ x',x''\to x}} \frac{\partial J}{\partial x} = G_{31}\left(G_{22}G_{33} - G_{32}G_{23}\right) + G_{32}\left(G_{31}G_{23} - G_{21}G_{33}\right) + G_{33}\left(G_{21}G_{32} - G_{31}G_{22}\right) = 0\,, \tag{68}$$

$$\lim_{\substack{t',t''\to t\\ x',x''\to x}} \frac{\partial J}{\partial t} = G_{21}\left(G_{22}G_{33} - G_{32}G_{23}\right) + G_{22}\left(G_{31}G_{23} - G_{21}G_{33}\right) + G_{23}\left(G_{21}G_{32} - G_{31}G_{22}\right) = 0\,. \tag{69}$$

At this point, we can calculate the limit derivatives of probability density (63) with respect to x and t:

$$\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial x} = -\frac{1}{J\,|J|}\frac{\partial J}{\partial x} p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3} + \frac{1}{|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}\frac{\partial \widehat{\alpha}_i}{\partial x} = \frac{1}{|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}\frac{\partial \widehat{\alpha}_i}{\partial x}\,, \tag{70}$$

$$\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial t} = -\frac{1}{J\,|J|}\frac{\partial J}{\partial t} p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3} + \frac{1}{|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}\frac{\partial \widehat{\alpha}_i}{\partial t} = \frac{1}{|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}\frac{\partial \widehat{\alpha}_i}{\partial t}\,, \tag{71}$$

where we have used Eqs. (67). From Eqs. (64) and (67) and definitions (57) and (58) it follows that

$$\frac{\partial \widehat{\alpha}_i}{\partial x} = -\frac{1}{J}H_{i1}c\,, \qquad \frac{\partial \widehat{\alpha}_i}{\partial t} = -\frac{1}{J}H_{i1}b\,. \tag{72}$$

Substituting these results back into Eqs. (70) and (71) we obtain

$$\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial x} = -\frac{c}{J\,|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}H_{i1}\,, \qquad \frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial t} = -\frac{b}{J\,|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}H_{i1}\,. \tag{73}$$

At the same time we notice that

$$-c\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a} = -\frac{c}{J\,|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}\frac{\partial \widehat{\alpha}_i}{\partial a} = -\frac{c}{J\,|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}H_{i1}\,, \tag{74}$$

$$-b\frac{\partial p^{(a,b,c)}_{\psi\psi_t\psi_x}}{\partial a} = -\frac{b}{J\,|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}\frac{\partial \widehat{\alpha}_i}{\partial a} = -\frac{b}{J\,|J|}\frac{\partial p^{(\widehat{\alpha}_1,\widehat{\alpha}_2,\widehat{\alpha}_3)}_{\xi_1\xi_2\xi_3}}{\partial \widehat{\alpha}_i}H_{i1}\,, \tag{75}$$

and, therefore, differential constraints (24) and (27) are satisfied identically [simply compare Eqs. (74) and (75) with Eqs. (73)]. Similarly, it is possible to show analytically that higher-order constraints such as Eqs. (33), (12), or (31) are satisfied identically by joint probability density function (63). We conclude this section by emphasizing that there exist many joint densities satisfying a single differential constraint [e.g., Eq. (12)], for the same boundary and initial conditions. As an example, simply consider the family of probability densities in the form of Eq. (63) where functions $\hat{\alpha}_i$ are translated by rather arbitrary functions $\chi_i(x', x'', t', t'')$ defined for $x', x'' \in \mathbb{R}$, $t', t'' \geq t_0$ and satisfying $\chi_i = 0$ for $t' = t_0$ or $t'' = t_0$. This clearly shows that a boundary value problem for Eq. (12) alone is ill posed; i.e., it admits an infinite number of solutions.

7. SUMMARY

We have determined and discussed new differential constraints satisfied by the joint probability density function of arbitrary random waves governed by Eq. (1) for random boundary conditions and random initial conditions. These differential constraints involve unusual limit partial differential operators, and they can be classified into two main groups: the first one depends on the stochastic wave equation while the second one includes a set of intrinsic relations determined by the structure of the joint probability density function of the wave and its derivatives. We have shown that in those cases where an evolution equation for the probability density function exists (e.g., for first-order nonlinear stochastic PDEs) the set of differential constraints is complete; i.e., it allows us to determine the probability density function of the solution. We have applied the new theory to a random wave process in a one-dimensional spatial domain and we have shown analytically that the differential constraints for the joint probability density function are satisfied identically. In addition, we have argued that a boundary value problem involving only one differential constraint is, in general, not well posed; i.e., it admits an infinite number of solutions. This brings us back to the fundamental question we have addressed at the beginning of this paper: How many differential constraints do we need in order to have enough information to compute the probability density function of the system uniquely? Most likely the full set of first-order differential constraints, which has been proven to be complete for first-order stochastic PDEs, is not sufficient and the full hierarchy of equations have to be considered (see, e.g., [40]). We conclude by emphasizing that the sequence of steps that led us to formulate the set of differential constraints presented in this paper can be repeated with some modifications for other linear and nonlinear stochastic PDEs. In other words, we have provided a general methodology that allows us to obtain an ensemble of conditions, in the form of partial differential equations, for the probability density function of the solution to an arbitrary stochastic PDE.

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APPENDIX A. FUNCTIONAL REPRESENTATION OF THE PROBABILITY DENSITY FUNCTION OF THE SOLUTION TO STOCHASTIC PDES

Let us consider a physical system described in terms of PDEs subject to uncertain initial conditions, boundary conditions, physical parameters, or external forcing terms. The solution to these types of boundary value problems is a random field whose regularity properties in space and time are related strongly to the type of nonlinearities appearing in the equations as well as to the statistical properties of the random input processes. In this paper we assume that the probability density function of the solution field exists. In order to fix ideas, let us consider the advection-diffusion equation

$$\frac{\partial\psi}{\partial t} + \xi(\omega)\frac{\partial\psi}{\partial x} = \nu\frac{\partial^2\psi}{\partial x^2}\,, \tag{A.1}$$

with deterministic boundary and initial conditions. The parameter $\xi(\omega)$ is assumed to be a random variable with known probability density function. The solution to Eq. (A.1) is a random field depending on the random variable $\xi(\omega)$ in a possibly nonlinear way. We shall denote such a functional dependence as $\psi(x,t;[\xi])$. The joint probability density of $\psi(x,t;[\xi])$ and ξ [i.e., the solution field at the space-time location (x,t) and the random variable ξ] admits the following integral representation (see Eq. (16) in [30]):

$$p^{(a,b)}_{\psi(x,t)\xi} = \langle \delta(a-\psi(x,t;[\xi]))\delta(b-\xi)\rangle\,, \quad a,b\in\mathbb{R}\,. \tag{A.2}$$

The average operator $\langle\cdot\rangle$, in this particular case, is defined as a simple integral with respect to the probability density of $\xi(\omega)$; i.e.,

$$p^{(a,b)}_{\psi(x,t)\xi} = \int_{-\infty}^{\infty} \delta[a-\psi(x,t;[z])]\delta(b-z)p^{(z)}_{\xi}dz\,, \tag{A.3}$$

where $p^{(z)}_{\xi}$ denotes the probability density of ξ which could be compactly supported (e.g., a uniform distribution). The support of the probability density function $p^{(a,b)}_{\psi(x,t)\xi}$ is determined by the nonlinear transformation $\xi \to \psi(x,t;[\xi])$ appearing within the delta function $\delta[a-\psi(x,t;[\xi])]$ (see, e.g., Chap. 3 in [31]). Simple representation (A.3) can be generalized easily to infinite dimensional random input processes. To this end, let us examine the case where the scalar field ψ is advected by a random velocity field U according to the equation

$$\frac{\partial\psi}{\partial t} + U(x,t;\omega)\frac{\partial\psi}{\partial x} = \nu\frac{\partial^2\psi}{\partial x^2}\,, \tag{A.4}$$

for some deterministic initial condition and boundary conditions. Disregarding the particular form of the random field $U(x,t;\omega)$, let us consider its collocation representation for a given discretization of the space-time domain. This gives us a certain number of random variables $\{U(x_i,t_j;\omega)\}$ ($i=1,..,N$, $j=1,...,M$). The random field ψ solving Eq. (A.4) at each one of these locations is, in general, a nonlinear function of all the variables $\{U(x_i,t_j;\omega)\}$. In order to see this, it is sufficient to write an explicit finite-difference numerical scheme of Eq. (A.4). The joint probability density of the solution field ψ at (x_i,t_j) and the driving field U at (x_n,t_m), admits the following integral representation:

$$p^{(a,b)}_{\psi(x_i,t_j)U(x_n,t_m)} = \langle \delta\{a-\psi(x_i,t_j;[U(x_1,t_1),...,U(x_N,t_M)])\}\delta[b-U(x_n,t_m)]\rangle\,, \tag{A.5}$$

where the average, in this case, is with respect to the joint probability of all the random variables $\{U(x_n,t_m;\omega)\}$ ($n=1,..,N$, $m=1,...,M$). The notation $\psi(x_i,t_j;[U(x_1,t_1),...,U(x_N,t_M)])$ emphasizes that the solution field $\psi(x_i,t_j)$ is, in general, a nonlinear function of all the random variables $\{U(x_n,t_m;\omega)\}$. If we send the number of these variables to infinity (i.e., we refine the space-time mesh to the continuum level), we obtain a functional integral representation of the joint probability density

$$p^{(a,b)}_{\psi(x,t)U(x',t')} = \langle \delta[a - \psi(x,t;[U])]\delta[b - U(x',t')]\rangle = \int \mathcal{D}[U]W[U]\delta[a - \psi(x,t;[U])]\delta[b - U(x',t')] \,, \quad \text{(A.6)}$$

where $W[U]$ is the probability density functional of random field $U(x,t;\omega)$ and $\mathcal{D}[U]$ is the usual functional integral measure [13–15]. Depending on the particular equation of motion, we will need to consider different joint probability density functions; e.g., the joint probability of a field and its derivatives with respect to space variables. The functional representation described above allows us to deal with these different situations in a very practical way. For instance, the joint probability density of a field $\psi(x,y,t)$ and its first-order spatial derivatives at different space-time locations can be represented as

$$p^{(a,b,c)}_{\psi(x,y,t)\psi_x(x',y',t')\psi_y(x'',y'',t'')} = \langle \delta[a - \psi(x,y,t)]\delta[b - \psi_x(x',y',t')]\delta[c - \psi_y(x'',y'',t'')]\rangle \,, \quad \text{(A.7)}$$

where, for notational convenience, we have denoted by $\psi_x \overset{\text{def}}{=} \partial\psi/\partial x$, $\psi_y \overset{\text{def}}{=} \partial\psi/\partial y$ and we have omitted the functional dependence on the random input variables within each field. Similarly, the joint probability of $\psi(x,y,t)$ at two different spatial locations is

$$p^{(a,b)}_{\psi(x,y,t)\psi(x',y',t)} = \langle \delta[a - \psi(x,y,t)]\delta[b - \psi(x',y',t)]\rangle \,. \quad \text{(A.8)}$$

In order to lighten the notation further, sometimes we will drop the subscripts indicating the space-time variables and write, for instance

$$p^{(a,b)}_{\psi\psi'_x} = \langle \delta[a - \psi(x,y,t)]\delta[b - \psi_x(x',y',t')]\rangle \,, \quad \text{(A.9)}$$

or even more compactly

$$p^{(a,b)}_{\psi\psi'_x} = \langle \delta(a - \psi)\delta(b - \psi'_x)\rangle \,. \quad \text{(A.10)}$$

A.1 Representation of Derivatives

The differentiation of the probability density function with respect to space and time variables involves generalized derivatives of the Dirac delta function and it can be carried out in a systematic manner. To this end, let us consider Eq. (A.2) and define the following linear functional:

$$\int_{-\infty}^{\infty} p^{(a)}_{\psi(x,t)}\rho(a)da = \left\langle \int_{-\infty}^{\infty} \delta[a - \psi(x,t)]\rho(a)da \right\rangle = \langle \rho(\psi)\rangle \,, \quad \text{(A.11)}$$

where $\rho(a)$ is a continuously differentiable and compactly supported function. A differentiation of Eq. (A.11) with respect to t gives

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial p^{(a)}_{\psi(x,t)}}{\partial t}\rho(a)da &= \langle \psi_t \frac{\partial\rho}{\partial\psi}\rangle = \left\langle \psi_t \int_{-\infty}^{\infty} \frac{\partial\rho}{\partial a}\delta[a - \psi(x,t)]da \right\rangle \\ &= \int_{-\infty}^{\infty} -\frac{\partial}{\partial a}\langle \psi_t\delta[a - \psi(x,t)]\rangle\rho(a)da \,. \end{aligned} \quad \text{(A.12)}$$

This equation holds for an arbitrary $\rho(a)$ and therefore we have the identity

$$\frac{\partial p^{(a)}_{\psi(x,t)}}{\partial t} = -\frac{\partial}{\partial a}\langle \delta\left[a - \psi(x,t)\right]\psi_t(x,t)\rangle \,. \quad \text{(A.13)}$$

Similarly,

$$\frac{\partial p^{(a)}_{\psi(x,t)}}{\partial x} = -\frac{\partial}{\partial a}\langle \delta\left[a - \psi(x,t)\right]\psi_x(x,t)\rangle \,. \quad \text{(A.14)}$$

Straightforward extensions of these results allow us to compute derivatives of joint probability density functions involving more fields; e.g., $\psi(x,t)$ and its first-order spatial derivative $\psi_x(x,t)$. For instance, we have

$$\frac{\partial p^{(a,b)}_{\psi\psi_x}}{\partial t} = -\frac{\partial}{\partial a}\langle \delta\left[a-\psi(x,t)\right]\delta[b-\psi_x(x,t)]\psi_t(x,t)\rangle - \frac{\partial}{\partial b}\langle \delta\left[a-\psi(x,t)\right]\delta[b-\psi_x(x,t)]\psi_{tx}(x,t)\rangle, \tag{A.15}$$

where $\psi_{tx} \stackrel{\text{def}}{=} \partial^2\psi/\partial t\partial x$.

A.2 Representation of Averages

In this section we determine important formulas to compute the average of a product between Dirac delta functions and various fields. To this end, let us first consider the average $\langle\delta(a-\psi)\psi\rangle$. By applying well-known properties of Dirac delta functions it can be shown that

$$\langle\delta(a-\psi)\psi\rangle = a p^{(a)}_{\psi(x,t)}\,. \tag{A.16}$$

This result is a multidimensional extension of the following trivial identity that holds for one random variable ξ (with probability density $p^{(z)}_{\xi}$) and a nonlinear function $g(\xi)$ (see, e.g., Ch. 3 of [31] or [30]):

$$\int_{-\infty}^{\infty}\delta[a-g(z)]g(z)p^{(z)}_{\xi}dz = \sum_n \frac{1}{|g'(\widehat{z}_n)|}\int_{-\infty}^{\infty}\delta(z-\widehat{z}_n)g(z)p^{(z)}_{\xi}dz = \sum_n \frac{g(\widehat{z}_n)p^{(\widehat{z}_n)}_{\xi}}{|g'(\widehat{z}_n)|}\,, \tag{A.17}$$

where $\widehat{z}_n = g^{-1}(a)$ are roots of $g(z)=a$. Since $g(\widehat{z}_n) = g(g^{-1}(a)) = a$, from Eq. (A.17) it follows that

$$\int_{-\infty}^{\infty}\delta[a-g(z)]g(z)p^{(z)}_{\xi}dz = a\sum_n \frac{p^{(\widehat{z}_n)}_{\xi}}{|g'(\widehat{z}_n)|} = a\int_{-\infty}^{\infty}\delta(a-g(z))p^{(z)}_{\xi}dz\,, \tag{A.18}$$

which is the one-dimensional version of Eq. (A.16). Similarly, one can show that

$$\langle\delta(a-\psi)\delta(b-\psi_x)\psi\rangle = a p^{(a,b)}_{\psi\psi_x}\,, \tag{A.19}$$

$$\langle\delta(a-\psi)\delta(b-\psi_x)\psi_x\rangle = b p^{(a,b)}_{\psi\psi_x}\,, \tag{A.20}$$

and, more generally, that

$$\langle\delta(a-\psi)\delta(b-\psi_x)h(x,t,\psi,\psi_x)\rangle = h(x,t,a,b)p^{(a,b)}_{\psi\psi_x}\,, \tag{A.21}$$

where, for the purposes of the present paper, $h(\psi,\psi_x,x,t)$ is any continuous function of ψ, ψ_x, x, and t. The result [Eq. (A.21)] can be generalized even further to averages involving a product of functions in the form

$$\begin{aligned}&\langle\delta(a-\psi)\delta(b-\psi_x)h(x,t,\psi,\psi_x)g(x,t,\psi_{xx},\psi_{xt},\psi_{tt},...)\rangle\\ &= h(x,t,a,b)\langle\delta(a-\psi)\delta(b-\psi_x)g(x,t,\psi_{xx},\psi_{xt},\psi_{tt},...)\rangle\,.\end{aligned} \tag{A.22}$$

In short, the general rule is: we are allowed to take out of the average all those functions involving fields for which we have available a Dirac delta. As an example, if ψ is a time-dependent random field in a two-dimensional spatial domain we have

$$\begin{aligned}&\langle\delta(a-\psi)\delta(b-\psi_x)\delta(c-\psi_y)e^{-(x^2+y^2)}\sin(\psi)\psi_x\psi_y^2\psi_{xx}\rangle\\ &= e^{-(x^2+y^2)}\sin(a)bc^2\langle\delta(a-\psi)\delta(b-\psi_x)\delta(c-\psi_y)\psi_{xx}\rangle\,.\end{aligned} \tag{A.23}$$

APPENDIX B. HOPF CHARACTERISTIC FUNCTIONAL APPROACH

In this appendix we employ a Hopf characteristic functional approach [19, 32–34] to derive the differential constraint (12) for the probability density function of one-dimensional random waves satisfying Eq. (2). This allows us to show how the differential constraints for the probability density function can be obtained from general principles. The joint Hopf characteristic functional of the wave ψ and its derivatives with respect to space and time is defined as

$$F[\alpha, \beta, \gamma] \stackrel{\text{def}}{=} \langle e^{\mathcal{Z}[\alpha,\beta,\gamma]} \rangle \,, \tag{B.1}$$

where

$$\begin{aligned} \mathcal{Z}[\alpha, \beta, \gamma] = i \int_T \int_X \psi\,(X, \tau; \omega)\, \alpha(X, \tau) dX d\tau + i \int_T \int_X \psi_t\,(X, \tau; \omega)\, \beta(X, \tau) dX d\tau \\ + i \int_T \int_X \psi_x\,(X, \tau; \omega)\, \gamma(X, \tau) dX d\tau \end{aligned} \tag{B.2}$$

and the average operator is a functional integral with respect to the joint probability functional of the random initial condition and the random boundary conditions. The Volterra functional derivative [35, 36] of Eq. (B.1) with respect to $\psi(x, t)$; i.e., the Gâteaux differential [37, 38] of $F[\alpha, \beta, \gamma]$ with respect to α evaluated at $z(X, t) = \delta(t - \tau)\delta(x - X)$, is

$$\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi(x, t)} \stackrel{\text{def}}{=} i \langle \psi(x, t; \omega) e^{\mathcal{Z}[\alpha,\beta,\gamma]} \rangle \,. \tag{B.3}$$

Similarly, the functional derivatives of Eq. (B.1) with respect to $\psi_t(x, t)$ and $\psi_x(x, t)$ are

$$\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi_t(x, t)} = i \langle \psi_t(x, t; \omega) e^{\mathcal{Z}[\alpha,\beta,\gamma]} \rangle \,, \tag{B.4}$$

$$\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi_x(x, t)} = i \langle \psi_x(x, t; \omega) e^{\mathcal{Z}[\alpha,\beta,\gamma]} \rangle \,. \tag{B.5}$$

Note that, by construction,

$$\frac{\partial}{\partial x}\left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi(x, t)}\right) = \frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi_x(x, t)} \,, \quad \text{and} \quad \frac{\partial}{\partial t}\left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi(x, t)}\right) = \frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi_t(x, t)} \,. \tag{B.6}$$

If we differentiate Eq. (B.3) twice with respect to t and x

$$\frac{\partial^2}{\partial t^2}\left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi(x, t)}\right) = i \langle \psi_{tt}(x, t; \omega) e^{\mathcal{Z}[\alpha,\beta,\gamma]} \rangle \,, \tag{B.7}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi(x, t)}\right) = i \langle \psi_{xx}(x, t; \omega) e^{\mathcal{Z}[\alpha,\beta,\gamma]} \rangle \,. \tag{B.8}$$

and we take into account Eq. (2), we easily determine the following functional differential equation governing for the dynamics of the Hopf characteristic functional:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi(x, t)}\right) = U^2 \frac{\partial^2}{\partial x^2}\left(\frac{\delta F[\alpha, \beta, \gamma]}{\delta\psi(x, t)}\right) \,. \tag{B.9}$$

We notice that up to this point we have made no use of the fields β and γ appearing in Eq. (B.1). Indeed, we can safely set $\beta = 0$ and $\gamma = 0$ in Eqs. (B.2) and (B.1) and determine exactly the same evolution [Eq. (B.9)]. The need for β and γ clearly appears when one tries to extract a pointwise equation for the probability density function of the random wave from Eq. (B.9). To this end, let us first remark that functional differential equation (B.9) holds for arbitrary test functions α, β, and γ. In particular, it holds for

$$\alpha^+(X, \tau) = a\delta(t - \tau)\delta(x - X) \,, \tag{B.10}$$

$$\beta^+(X,\tau) = b\delta(t'-\tau)\delta(x'-X)\,, \tag{B.11}$$

$$\gamma^+(X,\tau) = c\delta(t''-\tau)\delta(x''-X)\,. \tag{B.12}$$

An evaluation of Eq. (B.9) for $\alpha = \alpha^+$, $\beta = \beta^+$, and $\gamma = \gamma^+$ yields the condition

$$i\left\langle \left[\psi_{tt}(x,t;\omega) - U^2\psi_{xx}(x,t;\omega)\right] e^{i\psi(x,t;\omega)a + i\psi_t(x',t';\omega)b + i\psi_x(x'',t'';\omega)c}\right\rangle = 0\,. \tag{B.13}$$

This condition is equivalent to a differential constraint involving the joint characteristic function of the wave and its derivatives

$$\phi^{(a,b,c)}_{\psi\psi'_x\psi''_t} \stackrel{\text{def}}{=} \langle e^{i\psi(x,t;\omega)a + i\psi_t(x',t';\omega)b + i\psi_x(x'',t'';\omega)c}\rangle\,, \tag{B.14}$$

In order to see this, let us notice that

$$\begin{aligned}\frac{\partial^2\phi^{(a,b,c)}_{\psi\psi'_x\psi''_t}}{\partial t^2} &= ia\langle\psi_{tt}(x,t;\omega)e^{i\psi(x,t;\omega)a + i\psi_t(x',t';\omega)b + i\psi_x(x'',t'';\omega)c}\rangle \\ &\quad - a^2\langle\psi_t(x,t;\omega)^2 e^{i\psi(x,t;\omega)a + i\psi_t(x',t';\omega)b + i\psi_x(x'',t'';\omega)c}\rangle\,.\end{aligned} \tag{B.15}$$

If we take the limit of this expression for $(t',t'') \to t$ and $(x',x'') \to x$ we obtain

$$\frac{\mathfrak{d}^2\phi^{(a,b,c)}_{\psi\psi_x\psi_t}}{\mathfrak{d}t^2} = ia\langle\psi_{tt}(x,t;\omega)e^{i\psi(x,t;\omega)a + i\psi_t(x,t;\omega)b + i\psi_x(x,t;\omega)c}\rangle + a^2\frac{\partial^2\phi^{(a,b,c)}_{\psi\psi_x\psi_t}}{\partial b^2}\,, \tag{B.16}$$

where $\mathfrak{d}$ denotes the limit partial derivative operator. Similarly,

$$\frac{\mathfrak{d}^2\phi^{(a,b,c)}_{\psi\psi_x\psi_t}}{\mathfrak{d}x^2} = ia\langle\psi_{xx}(x,t;\omega)e^{i\psi(x,t;\omega)a + i\psi_x(x,t;\omega)b + i\psi_t(x,t;\omega)c}\rangle + a^2\frac{\partial^2\phi^{(a,b,c)}_{\psi\psi_x\psi_t}}{\partial c^2}\,. \tag{B.17}$$

At this point we can combine the results above to obtain the following differential constraint for the joint characteristic function:

$$\frac{\mathfrak{d}^2\phi^{(a,b,c)}_{\psi\psi_x\psi_t}}{\mathfrak{d}t^2} - a^2\frac{\partial^2\phi^{(a,b,c)}_{\psi\psi_x\psi_t}}{\partial b^2} = U^2\frac{\mathfrak{d}^2\phi^{(a,b,c)}_{\psi\psi_x\psi_t}}{\mathfrak{d}x^2} - U^2a^2\frac{\partial^2\phi^{(a,b,c)}_{\psi\psi_x\psi_t}}{\partial c^2}\,. \tag{B.18}$$

The inverse Fourier transform of this relation with respect to a, b, and c gives us exactly Eq. (12). In order to prove this, we simply recall the definition of $p^{(a,b,c)}_{\psi\psi_x\psi_t}$ as the inverse Fourier transform of the characteristic function $\phi^{(a,b,c)}_{\psi\psi_x\psi_t}$:

$$p^{(a,b,c)}_{\psi\psi_x\psi_t} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ia\alpha - ib\beta - ic\eta}\phi^{(\alpha,\beta,\eta)}_{\psi\psi_x\psi_t}d\alpha d\beta d\eta\,, \tag{B.19}$$

and two simple relations arising from Fourier transformation theory of a one dimensional function $f(x)$:

$$\int_{-\infty}^{\infty} e^{-iax}\frac{d^n f(x)}{dx^n}dx = (ia)^n\int_{-\infty}^{\infty} e^{-iax}f(x)dx\,, \tag{B.20}$$

$$\int_{-\infty}^{\infty} e^{-iax}x^n f(x)dx = i^n\frac{d^n}{da^n}\int_{-\infty}^{\infty} e^{-iax}f(x)dx\,. \tag{B.21}$$

APPENDIX C. PERTURBATION EXPANSIONS OF DIFFERENTIAL CONSTRAINTS

In this appendix we show that differential constraint (12) can be expanded in a perturbation series leading to an equation governing the local behavior of the probability density function in the neighborhood of the hyperplane $x'' = x' = x$, $t'' = t' = t$. To this end, let us consider again Eq. (8):

$$\begin{aligned}\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial t^2} - U^2\frac{\partial^2 p^{(a,b,c)}_{\psi\psi'_t\psi''_x}}{\partial x^2} &= \frac{\partial^2}{\partial a^2}\langle\delta\left(a-\psi\right)\psi_t^2\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle \\ &\quad + U\frac{\partial^2}{\partial a^2}\langle\delta\left(a-\psi\right)\psi_x^2\delta\left(b-\psi'_t\right)\delta\left(c-\psi''_x\right)\rangle\end{aligned} \tag{C.1}$$

and look for an approximation to both averages appearing on the right-hand side. The simplest one consists in expanding ψ_t^2 and ψ_x^2 in a first-order Taylor series around (x', t') and (x'', t''), respectively; i.e.,

$$\psi_t^2 = \psi_t'^2 + 2\psi_t'\psi_{tx}'(x - x') + 2\psi_t'\psi_{tt}'(t - t') + \cdots , \tag{C.2}$$

$$\psi_x^2 = \psi_x''^2 + 2\psi_x''\psi_{xx}''(x - x'') + 2\psi_x''\psi_{xt}''(t - t'') + \cdots . \tag{C.3}$$

A substitution of Eqs. (C.2) and (C.3) into the averages in Eq. (C.1) yields

$$\begin{aligned}&\langle \delta(a-\psi)\, \psi_t^2 \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle = \langle \delta(a-\psi)\, \psi_t'^2 \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle \\ &+2(x-x')\langle \delta(a-\psi)\, \psi_t'\psi_{tx}' \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle + 2(t-t')\langle \delta(a-\psi)\, \psi_t'\psi_{tt}' \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle ,\end{aligned} \tag{C.4}$$

$$\begin{aligned}&\langle \delta(a-\psi)\, \psi_x^2 \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle = \langle \delta(a-\psi)\, \psi_x''^2 \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle \\ &+2(x-x'')\langle \delta(a-\psi)\, \psi_x''\psi_{xx}'' \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle + 2(t-t'')\langle \delta(a-\psi)\, \psi_x''\psi_{xt}'' \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle .\end{aligned} \tag{C.5}$$

At this point we can use property (A.22) and write Eqs. (C.4) and (C.5) as

$$\begin{aligned}\langle \delta(a-\psi)\, \psi_t^2 \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle &= b^2 p^{(a,b,c)}_{\psi\psi_t'\psi_x''} + 2b(x-x') \int_{-\infty}^{b} \frac{\partial p^{(a,b',c)}_{\psi\psi_t'\psi_x''}}{\partial x'} db' \\ &+ 2b(t-t') \int_{-\infty}^{b} \frac{\partial p^{(a,b',c)}_{\psi\psi_t'\psi_x''}}{\partial t'} db' ,\end{aligned} \tag{C.6}$$

$$\begin{aligned}\langle \delta(a-\psi)\, \psi_x^2 \delta(b-\psi_t')\, \delta(c-\psi_x'')\rangle &= c^2 p^{(a,b,c)}_{\psi\psi_t'\psi_x''} + 2c(x-x'') \int_{-\infty}^{c} \frac{\partial p^{(a,b,c')}_{\psi\psi_t'\psi_x''}}{\partial x''} dc' \\ &+ 2c(t-t'') \int_{-\infty}^{c} \frac{\partial p^{(a,b,c')}_{\psi\psi_t'\psi_x''}}{\partial t''} dc' .\end{aligned} \tag{C.7}$$

A substitution of Eqs. (C.6) and (C.7) into Eq. (C.1) yields a fourth-order (nine-dimensional) partial differential equation that characterizes locally; i.e., in the neighborhood of the hyperplane $x = x' = x''$, $t = t' = t''$ the evolution of the probability density function of the random wave. Equivalently, we can say that we have determined a local (non-pointwise) realization of functional differential Eq. (B.9). An alternative method to obtain this result consists in evaluating functional differential Eq. (B.9) for test functions α, β, and γ that are very concentrated near specific space-time locations. These types of test functions can be any of those belonging to a delta sequence (see, e.g., Appendix A in [39]).