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UNCERTAINTY QUANTIFICATION OF THE GEM CHALLENGE MAGNETIC RECONNECTION PROBLEM USING THE MULTILEVEL MONTE CARLO METHOD

Éder M. Sousa,[1] Guang Lin,[2,*] & Uri Shumlak[1]

[1] Aerospace and Energetics Research Program, University of Washington, Seattle, Washington 98195, USA

[2] Department of Mathematics and School of Mechanical Engineering, Purdue University, West Lafayette, Indiana 47907, USA

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Plasma modelers often change the ion-to-electron mass ratio and speed of light to Alfvén speed ratio to decrease computational cost. Changing these parameters may affect the outcome of simulation and uncertainty in the results. This work aims to quantify the uncertainty of varying the ion-to-electron mass ratio, speed of light to Alfvén speed ratio, and the initial magnetic flux perturbation on the reconnected flux to provide a confidence limit. In this study, the multilevel Monte Carlo (MMC) method is employed to estimate the mean and variance, and the results are compared with the standard Monte Carlo (MC) and the probabilistic collocation (PC) methods. The plasma model used here is the two-fluid plasma where ions and electrons are treated as two separate fluids. Numerical simulations are presented showing the effectiveness of the MMC method when applied to the quasi-neutral ion cyclotron waves and the Geospace Environment Modeling (GEM) magnetic reconnection challenge problems. The mean reconnected flux with error bars provides a reconnection flux variation envelope, which can help numerical modelers to evaluate whether their reconnection flux lies inside the envelope for different plasma models. The results of the MMC mean and variance are comparable to the MC method but at a much lower computational cost.

KEY WORDS: *Probabilistic collocation, Monte Carlo, multi-level Monte Carlo, two-fluid plasma model, magnetic reconnection, quasi-neutral ion cyclotron wave*

1. BACKGROUND

There are numerous sources of uncertainty in many numerical plasma models, including the two-fluid plasma model (TFPM) [1–3]. These uncertainties range from individual parameters uncertainty to boundary or initial conditions, transport coefficients among others. Treating all the inputs as stochastic is computationally expensive. Therefore, different sensitivity analysis methods have been developed to rank the importance of the random inputs and their interactions.

The multilevel Monte Carlo (MMC) method was developed by Giles [4] for use in computational finance for the solution of stochastic differential equations. Giles proved that computational complexity could be reduced through the use of a multilevel approach that combines results obtained using different levels of discretization while reducing the variance. The MMC method uses a geometric sequence of time steps similar to the multigrid method and has proven to be efficient and reliable in achieving the desired accuracy.

The MMC method was extended from stochastic ordinary differential equations (ODEs) to elliptic partial differential equations (PDEs) in problems with random coefficients arising in ground water flow [5]. In the case of the PDEs the geometric series is applied to the spatial discretization instead of the temporal, although both may be related

*Correspond to Guang Lin, E-mail: guanglin@purdue.edu

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through the Courant, Friedrichs, and Lewy (CFL) stability condition. Here, there is a dramatic reduction in cost associated with the MMC method over the standard Monte Carlo (MC) method. This is because most of the uncertainty can be captured on a coarse spatial discretization, therefore reducing the need for numerous simulations on the finer and more computationally expensive discretizations.

Another method used in this study is the probabilistic collocation (PC) method. The PC method was first introduced by Tatang and McRae [6], and recently Xiu and Hesthaven [7] have used Lagrange polynomial interpolation to construct high-order stochastic collocation methods. The properties of PC method were extensively studied in the past 10 years. The errors of integrating or interpolating functions with Sobolev regularity were analyzed for Smolyak constructions based on one-dimensional nested Clenshaw-Curtis rules [8–10]. The degree of exactness of the Smolyak quadrature using Clenshaw-Curtis and Gaussian one-dimensional rules was investigated in [9]. The efficiency of Clenshaw-Curtis-based sparse grid stochastic collocation was demonstrated in comparison to other stochastic methods on an elliptic problem [7]. In 2003, Gerstner and Griebel [11] introduced the dimension-adaptive tensor product quadrature method. Sparse grid collocation schemes were applied to solving stochastic natural convection problems [12]. An adaptive hierarchical sparse grid collocation algorithm has been developed [13, 14].

Reference [15] describes a multi-element probabilistic collocation method (ME-PCM) that was developed to study the sensitivity of the TFPM model. The ME-PCM based sensitivity analysis method is efficient and can achieve fast convergence compared to standard MC based sensitivity analysis methods for cases with 20 – 100 random parameters.

In the research reported here, the MMC, MC, and PC methods will be applied to the GEM challenge magnetic reconnection [16] problem for sensitivity and uncertainty quantification study. Magnetic reconnection is a physical process that occurs in plasma by which the magnetic field lines tear and reconnect and in the process convert magnetic energy into kinetic energy. This phenomenon is present in virtually every plasma, whether in laboratory fusion research experiments, stars solar flares, or even in the Earth's magnetosphere.

Reference [16] presents multiple plasma models that are used in an effort to understand the process on magnetic reconnection in better detail. In ideal magnetohydrodynamics (MHD) the magnetic field lines are frozen into and move with the plasma; however, for magnetic reconnection to occur, the frozen-in flow constraint must be broken and this can only be done if there is some resistivity or two-fluid effects.

In a fluid description the frozen-in constraint is broken and the magnetic topology is allowed to change when the scale sizes in the system approach the electron skin depth $\delta_e = c/\omega_{pe}$, where c is the speed of light and ω_{pe} is the electron plasma frequency. Therefore, changing the electron mass or the speed of light affects this scale size [17].

The plasma model used here is the two-fluid model which models the electrons and ions as separate fluids and couples both fluids through the Maxwell's equations [18]. The numerical scheme used is the wave propagation method [19]. It is a class of Godunov methods, which rely on the solution of a Riemann problem at each cell interface while allowing for discontinuities at these interfaces. Second-order accuracy is achieved by applying flux limiters.

The main objective of this paper is to evaluate the benefits of the MMC method and quantify the uncertainty in the GEM challenge problem due to the choice of ion-to-electron mass ratio, speed of light to Alfvén speed ratio, and the magnetic flux perturbation that determines the dominant mode in reconnection, on the reconnected flux evolution. The remainder of the paper is organized in the following way: Section 2 presents the two-fluid plasma model for uncertainty quantification study. Section 3 introduces the standard Monte Carlo method which will be compared to the MMC described in Section 4. Section 5 shows the results obtained for the different uncertainties on the parameter mentioned above, and Section 6 summarizes the findings and provides additional discussion. The PC method is presented in the Appendix.

2. PLASMA MODELS FOR UNCERTAINTY QUANTIFICATION STUDY

In this work, two different plasma models are employed to quantify the uncertainty and compare the three uncertainty quantification (UQ) methods (MC, MMC, and PC). The first plasma model is the quasi-neutral ion cyclotron waves model. The problem models a quasi-neutral ion cyclotron wave, a dispersive wave, in a uniform plasma in a magnetic field that is constant in space and time. Like the two-fluid plasma model, which is next, the quasi-neutral ion cyclotron waves has imaginary eigenvalues of the source Jacobian. The equation system being solved is given as

$$\frac{\partial \rho}{\partial t} + \rho \frac{d\mathbf{u}}{dx} = 0, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{u}}{\partial x} \right) + \frac{\partial p}{\partial x} = nq\mathbf{u} \times \mathbf{B} = \rho \omega_c \mathbf{u} \times \hat{b}, \tag{2}$$

$$\frac{\partial \epsilon}{\partial t} + \frac{d}{dx} \left((\epsilon + p) \mathbf{u} \right) = 0, \tag{3}$$

where ρ is the mass density, $\mathbf{u}$ is the velocity, p is the pressure, $\mathbf{B}$ is the magnetic field, $\hat{b}$ is the unit vector in the direction of $\mathbf{B}$, $\omega_c = q|\mathbf{B}|/m$ is the ion cyclotron frequency, and

$$\epsilon = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \tag{4}$$

is the total energy density, where γ is the ratio of specific heats.

A benefit of using this problem is that the exact solution of the velocity field is known and given by

$$u(x,t) = -\sum_{n=0}^{N} \frac{u_1^o}{2n+1} \sin(k_n x + \omega_n t), \tag{5}$$

when the fluid is initially perturbed with the random velocity (to simulate uncertainty),

$$u_1(x) = u_1^o(\xi) \sum_{n=0}^{N} \frac{i}{2n+1} e^{ik_n x}, \tag{6}$$

where $u_1^o(\xi)$ is the random velocity, ξ is a random variable with uniform distribution, $U[10^{-10}$ to $10^{-4}]$, $k_n = 2\pi(2n+1)$ is the wave number, $\omega_n = \pm\sqrt{k_n^2 c_s^2 + \omega_c^2}$ is the dispersion relation, and $c_s = \sqrt{\gamma p/\rho}$ is the speed of sound. This problem is computationally inexpensive.

The second plasma model is the collisionless TFPM [1–3], which is derived by taking moments of the Boltzmann equation and regarding the electron and ion species as separate fluids. The resulting equation system for the TFPM is used to evolve the fluid variables, and Maxwell's equations advance the electric and magnetic fields. The fluids and fields variables couple through the source terms.

There are numerous sources of uncertainty in the two-fluid plasma model. Often times modelers change the ion-to-electron mass ratio or the speed of light to Alfvén speed ratio to speed up the computation. This study aims to quantify the uncertainty of varying the ion-to-electron mass ratio, speed of light to Alfvén speed ratio, and the magnetic flux perturbation that determines the dominant mode of the reconnection, have on the reconnected flux of the the GEM magnetic reconnection challenge. In particular, we assume $m_i/m_e = \xi_1$, where ξ_1 is a random variable with uniform distribution $U[25, 100]$, $v_A/c = \xi_2$, ξ_2 is a random variable with uniform distribution, $U[10, 20]$, $v_A = B/\sqrt{\mu_o \rho}$ and the magnetic flux perturbation, $\psi = \xi_3 \psi_p$. Here ξ_3 is a random variable with uniform distribution, $U[0.085, 0.115]$.

The balance laws for the two-plasma fluids are given by

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{u}_s) = 0, \tag{7}$$

$$\frac{\partial \rho_s \mathbf{u}_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s + p_s \mathbf{I}) = \frac{\rho_s q_s}{m_s} (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}), \tag{8}$$

$$\frac{\partial \epsilon_s}{\partial t} + \nabla \cdot ((\epsilon_s + p_s) \mathbf{u}_s) = \frac{\rho_s q_s}{m_s} \mathbf{u}_s \cdot \mathbf{E}, \tag{9}$$

where the subscript s denotes the ion/electron species, ρ is the density, $\mathbf{u}$ is the velocity, p is the pressure, q is the charge, m is the mass, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively, and ϵ is the total energy density defined by Eq. (4). Maxwell's equations are expressed as

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \tag{10}$$

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_o \sum_s \frac{q_s}{m_s} \rho_s \mathbf{u}_s, \tag{11}$$

$$\epsilon_o \nabla \cdot \mathbf{E} = \sum_s \frac{q_s}{m_s} \rho_s, \tag{12}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{13}$$

The source terms in Maxwell's equations couple the fluid variables to the electromagnetic fields and the Lorentz forces act as body forces on the electrons and ions. These source terms introduce physical dispersion in the two-fluid model, which comes from the fact that the flux Jacobian of the system has imaginary eigenvalues and therefore the waves in the system are undamped [3].

Contrary to the MHD model, the two-fluid plasma model includes the Hall and diamagnetic-drift terms, assumes a finite speed of light, and a finite electron mass. These assumptions allow for the description of displacement current effects and high-frequency oscillations.

Both models are solved using the high-resolution wave propagation (a finite volume) method. The solution method belongs to the class of Godunov solvers, which rely on the solution of Riemann problems described in [1, 19, 20] and can be applied to balance laws of the form

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathcal{F} = \mathbf{S}, \tag{14}$$

where $\mathbf{Q}$ represents the conserved variables, $\mathcal{F}$ is the flux tensor, and $\mathbf{S}$ is the source term. A Riemann problem is solved at each cell interface and the solution is, in general, discontinuous. Second-order accuracy can be achieved by performing a linear reconstruction of the waves needed to compute the numerical fluxes at the cell interfaces [20].

3. STANDARD MONTE CARLO METHOD

In statistics, aggregate mathematical measures are created to characterize data; these rules are referred to as estimators. Common estimators are the mean value and the variance of the data set. In the standard MC method the mean estimator is given as

$$\mathbb{E}[P_{m,N}^{\text{MC}}] = \frac{1}{N}\sum_{i=1}^{N} P_m^i, \tag{15}$$

where N is the number of samples and P_m^i is a measured quantity of each sample i obtained using m spatial grid points. The value of P_m^i obtained using m grid points approaches the exact solution, P, as the number of grid points increases and the mean square error (MSE), e^2, can be represented as

$$e_{\text{MC}}^2 = \mathbb{V}[P_{m,N}^{\text{MC}}] + (\mathbb{E}[P_m] - \mathbb{E}[P])^2. \tag{16}$$

The first term represents the sampling error and is given by

$$\mathbb{V}[P_{m,N}^{\text{MC}}] = \frac{1}{N-1}\sum_{i=1}^{N}(P_m^i - \mathbb{E}[P_{m,N}^{\text{MC}}])^2, \tag{17}$$

which is the variance. The second term of Eq. (16) represents the discretization error. To achieve an MSE less than ϵ^2, both terms are chosen to be less that $\epsilon^2/2$. Reducing the first term is achieved by having a large number of samples, $N \sim O(\epsilon^{-2})$. The second term is reduced by increasing the number of grid points (increased resolution), $m \sim O(\epsilon^{-1/\alpha})$ where α is the discretization convergence rate [5].

4. MULTILEVEL MONTE CARLO METHOD

For most numerical simulations, either large sample size or high discretization are computationally expensive. Therefore, novel methods that reduce the numerical error while achieving "reasonable" discretization and sample size are desired. In this case the multilevel Monte Carlo (MMC) method is chosen to quantify the uncertainty in the two-fluid plasma model.

When there is uncertainty in a quantity or constant being used in a numerical simulation it is common to use a Monte Carlo method to study the effects of this uncertainty in the solution of the problem. This is done by running a large number of simulations each using a randomly selected value of set quantity within the uncertainty interval. All the simulations are executed using the same discretization. For the MMC method for each random quantity two simulations are done at different discretization.

The MMC estimator is defined as

$$P_m^{\text{MMC}} = \sum_{l=0}^{L} Y_l, \tag{18}$$

where

$$Y_l = \frac{1}{N_l} \sum_{i=1}^{N_l} (P^i_{m_l} - P^i_{m_{l-1}}), \tag{19}$$

The MMC mean estimator is

$$\mathbb{E}[P_m^{\text{MMC}}] = \sum_{l=0}^{L} \mathbb{E}[Y_l], \tag{20}$$

where $P^i_{m_{-1}} = 0$. In this case, because there are different discretization levels, l, the spacial resolution becomes, m_l, to denote the number of grid points at level l. Also, the sample size (N) from Eq. (15) is now level dependent (N_l) and does not need to be the same for every level. Y_l is estimated independently for each level l and the multilevel variance estimator is

$$\mathbb{V}[P_m^{\text{MMC}}] = \sum_{l=0}^{L} \mathbb{V}[Y_l]. \tag{21}$$

Following Eq. (16) the error in the MMC method is expressed as

$$e^2_{\text{MMC}} = \sum_{l=0}^{L} \mathbb{V}[Y_l] + (\mathbb{E}[P_{m_l} - P])^2, \tag{22}$$

where

$$\mathbb{V}[Y_l] = \frac{1}{N_l} V(P_{m_l} - P_{m_{l-1}}). \tag{23}$$

The multilevel variance $\mathbb{V}[Y_l] = N_l^{-1} V(P_{m_l} - P_{m_{l-1}}) \to 0$ as $l \to \infty$ and if the variance is decreasing, a lesser number of sample data will be needed, consequently $N_l \to 1$ as $l \to \infty$. The cost at the coarsest level is fixed for all levels of accuracy. Achieving an overall MSE of ϵ with MMC method is easier than the standard MC method. The MMC method is used to measure the effect uncertainty in the initial velocity for the quasi-neutral ion cyclotron waves problem. This problem is chosen because it is computationally inexpensive. The MC and MMC methods are compared for numerical cost and variance in the solution. Then, the uncertainty due to the ion to electron mass ratio, light speed to Alfvén speed ratio, and initial magnetic flux perturbation in the collisionless magnetic reconnection problem are each studied separately. The reconnected flux for each case is computed for comparison.

Once the variable with uncertainty is identified, a range and distribution for the uncertainty are chosen. A uniform distribution is used in all cases in this study; however, other distributions could also be used (e.g., Maxwellian). For each simulation the specified variable is set to a random value within the given range. Since the distribution is uniform, every value has an equal probability of being selected.

The procedure for the multilevel method used in this implementation is described below.

- Start at level zero, $l = 0$, the coarsest grid.
- Step 1: Collect initial sample (velocity at $x = 0$ in the quasi-neutral ion cyclotron waves problem and magnetic field in the reconnection problem) at the current level and obtain initial statistics of the problem, mean and variance.
  - Calculate initial samples, each using a random coefficient (e.g., mass ratio, initial velocity...) from a uniform distribution. The resolution (spacial discretization) is given by $m_l = M^l m_o$, where M^l is the refinement multiplier (in this case $M = 2$, double grid points from level $l - 1$ to l), and m_o is the coarsest level resolution.
  - If the current level is zero, the simulations are standard Monte Carlo (the mean is calculated from Eq. (15)), and data are collected only at one discretization level.
  - If the current level is not zero, for each random coefficient, two cases are run, one at resolution $M^l m_o$ and another at the immediately lower resolution ($M^{l-1} m_o$). Then compute the difference between a chosen observed variable/quantity (e.g., velocity or reconnected flux) of the two cases $P_{m_l} - P_{m_{l-1}}$. Because the solutions are at different resolutions, the intermediate values at the coarse grid are the average of the values of the two closest grid-points.
  - The multilevel estimator of the mean of the solution is given by Eq. (20) and is calculated from data collected at two different discretization levels.
- Step 2: Estimate the variance using the initial sample. The unbiased variance of each level is given by Eq. (23), not to be confused with the overall method variance given in Eq. (21).
- Step 3: Calculate the optimal sample size for all levels up to the current one. The optimal number, N_l required to reduce the variance V_l to less than $(1/2)\epsilon^2$ is given from [4] as

$$N_l = 2\epsilon^{-2}\sqrt{V_l m_l}\left(\sum_{k=0}^{l}\sqrt{V_k/m_k}\right). \tag{24}$$

- Step 4: Collect additional sample at each level as needed for the new N_l. Run additional cases if N_l is bigger than the initial number of the sample.
- Step 5: Check for convergence. If $l \geq 1$, and $\max\left(M^{-1}|\hat{P}_{L-1}^{\text{MMC}}|, |\hat{P}_{L}^{\text{MMC}}|\right) < 1/\sqrt{2}(M-1)\epsilon$ then the error is less than $(1/\sqrt{2})\epsilon$ and the solution has converged.
- Step 6: If the solution has not converged, increment level to $l + 1$ and repeat from step 1.

Another uncertainty quantification method, probabilistic collocation, is described in the Appendix.

5. NUMERICAL RESULTS

The quasi-linear ion cyclotron wave problem is used to examine the performance of the MMC method in computing the expected values of the velocity. This model is nonlinear and less computationally expensive than the full two-fluid model. It is a one-dimensional (1D) problem, with periodic boundary conditions and the velocity field is initialized to the approximate step function given by Eq. (6) with $N = 9$. The results are analyzed after one unit of normalized time. The uncertainty is on u_1^o, Eq. (6), using a uniform distribution from 10^{-10} to 10^{-4}.

Figure 1 shows the L_2-norm error of the mean and the variance as a function of computational time. The L_2-norm error is given as

$$||\Delta u||_2 = \sqrt{\frac{1}{m_l}\sum(\hat{u} - u_i)^2}, \tag{25}$$

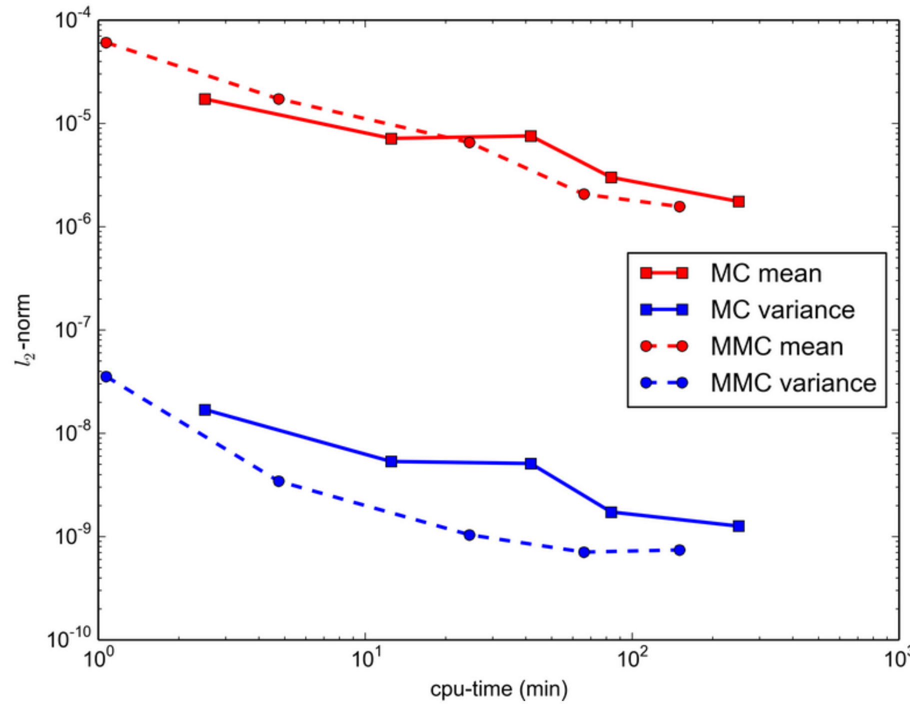

FIG. 1: L_2-norm error for the mean and variance as a function of computational time using the quasi-neutral ion cyclotron wave test problem, with $c_s = \sqrt{1.4}$, $\omega_c = 10.0$, $B_z = 1.0$, $m_i = 1.0$, and $q = 10.0$ and initial values of $\rho = 1.0$ and $p = 1.0$. The rates of convergence for the mean using the MC and MMC methods are similar. The rate of convergence for the variance using the MMC method is faster than the MC method.

where $\hat{u}$ is a highly converged solution computed using 30,000 MC samples, treated as the accurate solution. The mean and variance are calculated using smaller sample sizes using MC or MMC method and compared to converged solution. The slope of the MC variance line is about –0.561, while the MMC variance slope is –0.777.

The Geospace Environmental Modeling (GEM) reconnection challenge [16] is an effort to understand the mechanism of collisionless reconnection using different plasma models. The reconnection process tears magnetic fields and converts magnetic energy into kinetic and thermal energies of electrons and ions. Magnetic reconnection is present in numerous plasma environments including solar flares and geomagnetic sub-storms. The GEM reconnection challenge involves using different plasma models, including Hall-MHD, full particle, and hybrid models.

The initial conditions used for all cases are given in [16]. A Harris equilibrium is used with a floor in the density outside the current layer. The magnetic field is initialized as

$$B_x(y) = B_o \tanh(y/\lambda), \tag{26}$$

and the density as

$$n(y) = n_o \,\mathrm{sech}^2(y/\lambda) + n_\infty. \tag{27}$$

The pressure balance is given as $n_o(T_e + T_i) = B_o^2/8\pi$. The velocities are normalized to the Alfvén speed v_A. The normalized parameters for these simulations $B_o = 1$, $n_o = 1$, $\lambda = 0.5$, $n_\infty/n_o = 0.2$, $T_e/T_i = 0.2$. An initial perturbation in the magnetic flux is given as

$$\psi(x, y) = \psi_o \cos(2\pi x/L_x)\cos(\pi y/L_y), \tag{28}$$

and the magnetic field perturbation is given as $\mathbf{B} = \hat{z} \times \nabla\psi$ and $\psi_o = \xi_3$ which is the third source of uncertainty that is investigated in this research. When the uncertainties of m_i/m_e and c/v_A are being studied, ξ_3 is set to 0.1.

The unit scale length is the ion-skin depth (c/ω_{pi}) and the unit time scale is the inverse ion cyclotron frequency (ω_{ci}^{-1}). The computational domain is $L_x = 8\pi$ and $L_y = 4\pi$ and the final time is $400/\omega_{ci}$. The problem has periodic boundary conditions in the x-direction and conducting boundary conditions in the y-direction. To compare results, the reconnected flux, ϕ, is computed using

$$\phi(t) = \frac{1}{2L_x} \int_{-L_x/2}^{L_x/2} |B_y(x, y = 0, t)| dx. \tag{29}$$

The two-fluid plasma method used in these simulations is in good agreement with the benchmarked results published in the GEM challenge paper [16] and can be seen in [20].

For magnetic reconnection to occur and magnetic fields to tear and reconnect, the magnetic field lines cannot be frozen into the plasma and must move with the plasma. In a fluid description of plasmas the frozen-in constraint is broken and the magnetic topology is allowed to change when the length scales in the problem approach the electron skin depth $\delta_e = c/\omega_{pe}$, where c is the speed of light and ω_{pe} is the electron plasma frequency. In this study the effects of varying the speed of light and the electron mass are analyzed. Changing these quantities changes the electron skin depth, which in turn could affect the reconnected flux.

In a hydrogen plasma the ion-to-electron mass ratio is 1836. However, running simulations at this mass ratio presents a challenge. The two-fluid plasma model has disparate timescales, i.e., ω_{cs}^{-1} and ω_{ps}^{-1} need to be resolved, where

$$\omega_{cs} = \frac{q_s|\mathbf{B}|}{m_s}, \qquad \omega_{ps} = \sqrt{\frac{n_s q_s^2}{\epsilon_o m_s}}, \tag{30}$$

are the cyclotron frequency and the plasma frequency for each species, respectively. Decreasing the ion-to-electron mass ratio relaxes the time step restrictions and allows simulations to be completed in less time. A small artificial mass ratio of $m_i/m_e = 25$ is used in the GEM challenge paper to reduce the stiffness and speedup the computation.

Figure 2 shows the mean and variance reconnected flux for simulations by varying the mass ratio $m_i/m_e \in [25, 100]$ and the results of the MMC method is compared to the MC and PC methods results. The MMC simulations are completed using two levels with 32 simulations at a coarse grid (256×128) and 16 at a twice finer grid. The PC method has 33 simulations and the MC has 56, all at 512×256 resolution. The results for this case show that all three methods produce approximately the same mean and variance, σ^2.

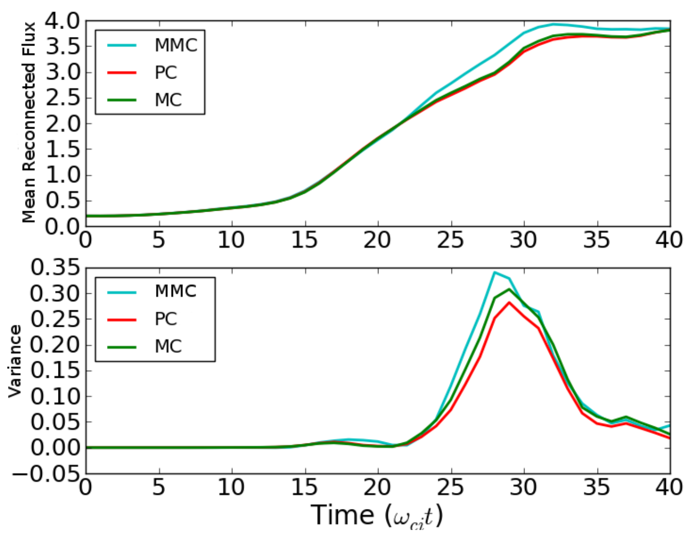

FIG. 2: Mean (top) and variance (bottom) of the reconnected flux for varying ion-to-electron mass ratio ranging from 25 to 100, for the MMC, MC, and PC methods.

For the MMC method the variance is computed using

$$\sigma^2[P_l] = \mathbb{E}[P_l^2] - \mathbb{E}^2[P_l], \tag{31}$$

where

$$\mathbb{E}[P_l^2] = \mathbb{E}[P_o^2] + \sum_{l=1}^{L} \mathbb{E}[P_l^2 - P_{l-1}^2] \tag{32}$$

instead of the multilevel estimator given by Eq. (21).

The second variable of the skin depth that is analyzed is the speed of light, c. In the two-fluid plasma model, explicitly advancing the equations in time is limited by the highest frequency or the fastest velocity in the system. The speed of light is often the fastest speed in the system and hence usually generates the most restrictive time step constraint. Relaxing this restriction can be achieved by reducing the value of c. As mentioned before, all the velocities in the GEM challenge paper are normalized to the Alfvén speed and in all simulations $\epsilon_o = \mu_o = 1$.

Figure 3 shows the mean and variance when the ratio of the speed of light to the Alfvén speed is varied from $c/v_A \in [10, 20]$ for the three different methods, MMC (2 levels, 32 samples at level 0 and 16 at level 1), MC (56 samples), and PC method (33 samples), at the resolution of 512×256 (for the MMC this is the resolution of level 1, finest grid resolution).

The GEM reconnection challenge Harris equilibrium is perturbed by a sinusoidal flux with an amplitude ψ_o as described by Eq. (28). Depending on the amplitude of this perturbation the reconnection is either delayed or sped up; therefore, the reconnected flux should be highly sensitive to variations in this parameter. Figure 4 shows the mean and variance of the reconnected flux when $\psi_o \in [0.085, 0.115]$ of the background magnetic field amplitude.

6. SUMMARY AND DISCUSSION

The MMC, MC, and PC methods are applied to the GEM magnetic reconnection problem to study the sensitivity of the reconnected flux to uncertainties in ion-to-electron mass ratio, speed of light to Alfvén speed ratio, and the magnitude of the initial magnetic flux perturbation. The goal is to determine which of the methods is more computational inexpensive and reduces the variance more rapidly as the sample size is increased.

Initially the performance of the different methods is studied using the quasi-linear ion cyclotron wave problem since it is computationally inexpensive. Figure 1 shows that the error decreases as the computational cost increases.

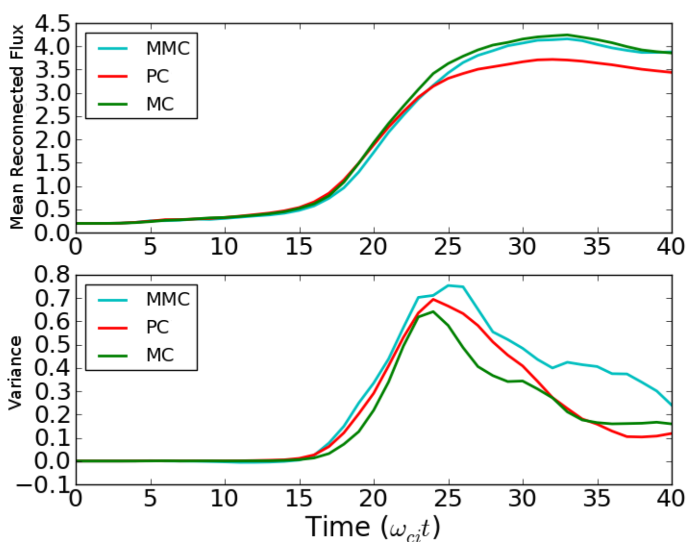

FIG. 3: Mean (top) and variance (bottom) of the reconnected flux for varying speed of light to Alfvén speed ratio ranging from 10 to 20, for the MMC, standard MC, and PC method.

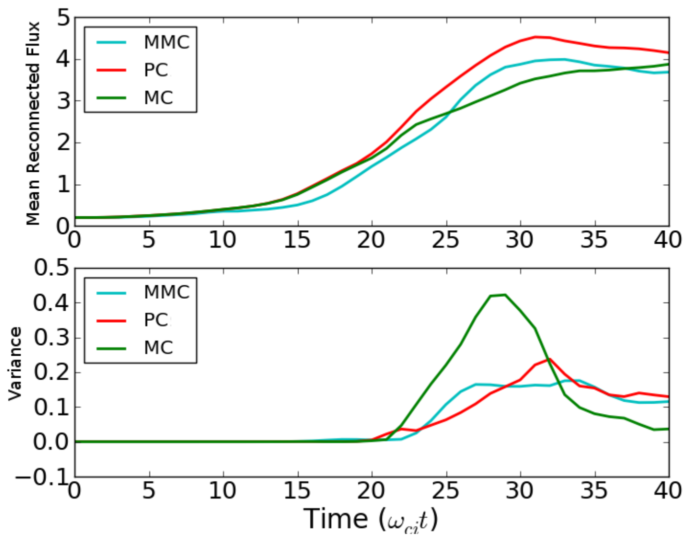

FIG. 4: Mean (top) and variance (bottom) of the reconnected flux for varying amplitude of the magnetic flux perturbation ranging from $[0.085, 0.115]$, for the MMC, standard MC, and PC method.

However, the error in the MMC variance is lower than the MC method, and the variance decreases much faster for the MMC method. Therefore the MMC method gives a better result with less computation effort.

Figure 2 shows the results for the mass ratio study. Here the variance is relatively low except between $\omega_{ci}t = 25$ – 35; therefore, the reconnected flux is relatively independent of the mass ratio. A possible explanation for the higher variance between $\omega_{ci}t = 25$ – 35 is that for certain cases an intermediate, smaller magnetic island forms at the center of the domain as seen in Fig. 5. This smaller island eventually merges with the bigger island.

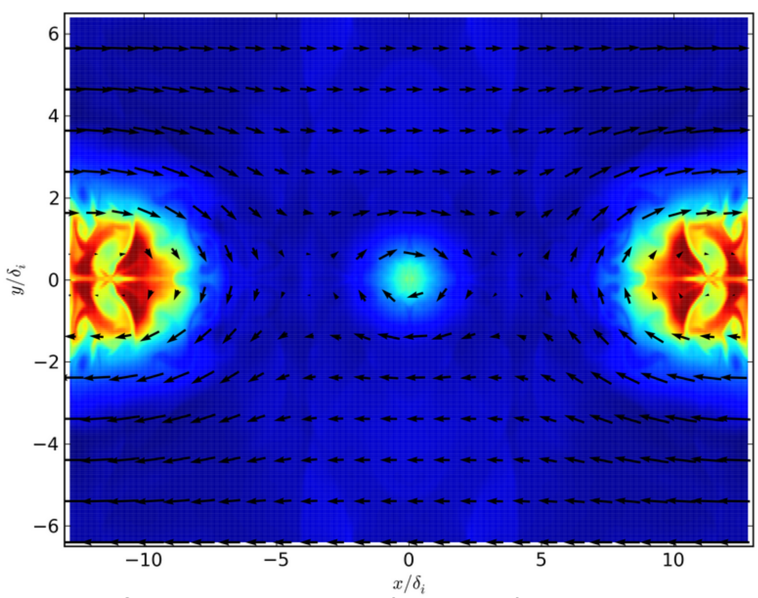

FIG. 5: Ion density and magnetic field vectors showing the formation of a second smaller magnetic island at the center and as a result the reconnected flux in these cases is slightly higher because there are no asymmetries in the grid and during the solution the smaller island will remain at the center of the grid. However, if some numerical asymmetry develops the island will move to the right or the left, $\omega_{ci}t = 38$ and $m_i/m_e = 25$.

Figure 3 demonstrates good agreement of the mean and the variance between the different methods showing the small sensitivity of the problem to changes in the speed of light to Alfvén speed ratio. Like the mass ratio case, the variance also spikes around $\omega_{ci}t = 23.0$ for the same reason as explained for the mass ratio.

The variance of the MC method is observed to be stochastically higher than the variance of the MMC and PC methods, as seen in Fig. 4. The MMC mean follows closely the MC mean. The cost of the MMC is much lower than the MC. The MMC method has 36 simulations at 256×128 resolution and only 6 simulations at 512×256 which resulted in a total of 4075 CPU-hours while the MC has 46 simulations at 512×256 resolution and a total of 6496 CPU-hours. Therefore, there is a total computational time saving of about one-third (37.3%) of the MMC method over the standard MC method and a smaller variance. The PC method has a total CPU time of 4660 CPU-hours (33 simulations at 512×128 resolution), which is 14.4% more computationally expensive as the MMC.

Figure 6 compares the reconnected flux computed by the different plasma models given in [16] and the reconnected flux computed by the two-fluid plasma model. The MHD and Hall-MHD models assume the speed of light is infinite while the hybrid, full particle, and two-fluid models assume a finite value. In the two-fluid case the standard deviation is plotted as the error bars to see the influence of varying the speed of light on the reconnected flux. Our results seem to be somewhere between the hybrid and the Hall-MHD models. The two-fluid plasma model results compare favorably with the full particle, Hall-MHD, and hybrid models. MHD has insufficient physics to capture collisionless reconnection.

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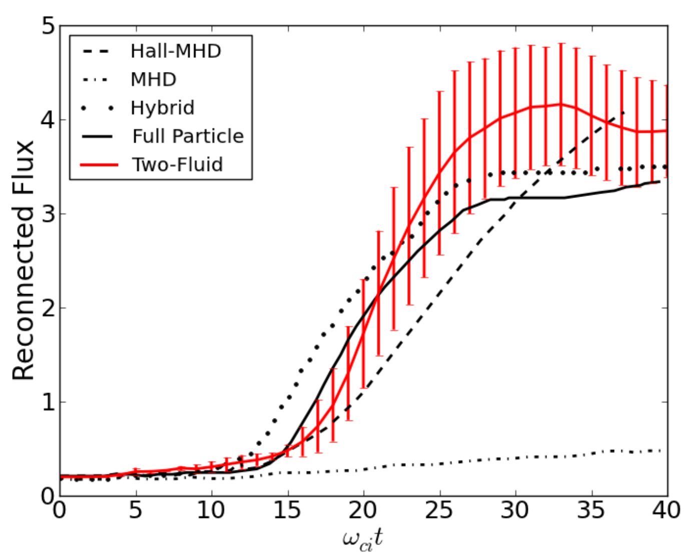

FIG. 6: Reconnected flux plotted versus time for different plasma models used in the GEM challenge paper [16] compared to two-fluid model. The error bars represent the uncertainty due to the sensitivity in the c/v_A as measured using the MMC method in Fig. 3.

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APPENDIX: PROBABILISTIC COLLOCATION METHOD

We compare the efficiency of the MMC method with other uncertainty quantification methods. In particular, we compare MMC with MC and PC methods. The general procedure for the PC method is similar to that of the MC method, except that different sampling points and corresponding weights are selected. The procedure consists of three main steps:

1. Generate N_c collocation points in a complete probability space, $(\Omega, \mathcal{A}, \mathcal{P})$, based on a quadrature formula based on the distribution of the random parameters (see details in [21, 22]), where Ω is the event space, $\mathcal{A} \in 2^{\Omega}$ the σ-algebra, and $\mathcal{P}$ the probability measure;

2. Solve a deterministic problem at each collocation point;

3. Estimate the solution statistics using the corresponding quadrature rule;

$$\begin{aligned}
\mathbb{E}[P(\boldsymbol{x}, t)] &= \int_{\Gamma} P(\boldsymbol{x}, t, \xi)\rho(\xi) d\xi \\
&\approx \sum_{k=1}^{N_c} P(\boldsymbol{x}, t, \xi_k)\rho(\xi_k) w_k, \qquad \text{(A.1)}\\
V[P](\boldsymbol{x}, t) &= \int_{\Gamma} (P(\boldsymbol{x}, t, \xi) - E[u])^2 \rho(\xi) d\xi \\
&\approx \sum_{k=1}^{N_c} P^2(\boldsymbol{x}, t, \xi_k)\rho(\xi) w_k - E[v]^2, \qquad \text{(A.2)}
\end{aligned}$$

where $\boldsymbol{x}$, t, and ξ are the spatial coordinate vector, time and random variable. $\rho(\xi)$ is the probabilistic distribution function (PDF) of random variable ξ, N_c is the number of quadrature points, $\{\xi_k\}$ is the set of quadrature points, and $\{w_k\}$ is the corresponding set of weights, which are the combination of quadrature weights in each random dimension. In the second step of the PC method approach, as for the MC method, any existing deterministic code can be used. Extensive reviews on the construction of quadrature formulas may be found in [21, 22]. In this work, the Smolyak formula [23] is used to construct the collocation point set, which is a linear combination of tensor product formulas and has a significantly smaller number of points than the full tensor product set. Recently, high-order stochastic collocation methods [7, 24, 25] has been developed based on sparse grids using the Smolyak formula [23]. Such sparse grids do not depend as strongly on the dimensionality of the random space and as such are more suitable for applications with high-dimensional random inputs. Detailed descriptions on building the collocation point set can be found in [7, 24, 25].