ELECTROCONVECTION INSTABILITY OF POORLY CONDUCTING FLUID IN ALTERNATING ELECTRIC FIELD

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The flat horizontal layer of the poorly conducting fluid is placed in the alternating electric field and heated from above. Its behavior is investigated in the electroconvection low-mode model. The approximation in which density and conductivity of the fluid are linearly dependent on temperature is used. Linear instability is analyzed by means of the Floquet theory. The system of eight differential equations, which describe the motion of the fluid, is integrated using the Runge-Kutta-Merson fourth-order method. The marginal stability curves are plotted in coordinates “wave number – nondimensional electric parameter.” The critical values of the wave number and the nondimensional electric parameter are determined for various external influence frequencies. The nonlinear regimes of the fluid flow are investigated at the critical value of the wave number. The fluid electroconvection flow intensity as a function of the nondimensional electric parameter is plotted. The various types of the oscillation regimes are discovered, and the competition regions of different electroconvection modes with various flow intensities are found.

KEY WORDS: electroconvection, low-mode model, electroconductive charge formation mechanism, oscillation regimes, alternating electric field

1. INTRODUCTION

The alternating electric field may cause different types of flow regimes in electrohydrodynamic systems (Bologa et al., 1977; Ostroumov, 1979; Stishkov and Ostapenko, 1989). Heat and mass transfer can be controlled by changing electric field amplitude and frequency, which is relevant in different technological situations.

The Lorenz model is often used to analyze flows in poorly conducting media (Berge et al., 1986; Lorenz, 1963). The Lorenz model was modified by adding complementary terms and variable coefficients in order to account for complicating factors. The behavior of fluids was investigated within this approach by Il’in and Kartavykh (2018), Il’in and Smorodin (2007), and Kartavykh et al. (2015).

Investigations of poorly conducting media are complicated because different charge formation mechanisms exist in fluids. The injective mechanism, the dielectrophoretic mechanism, and the electroconductive mechanism are the main ones of this type. The injective mechanism is associated with occurrence of charge on the fluid-electrode bound because of reduction-oxidation reactions (Atten and Lacroix, 1978; Lacroix et al., 1975; Smorodin and Taraut, 2014; Taraut and Smorodin, 2012; Zhakin, 2015). The dielectrophoretic mechanism of electroconvection is associated with the dependence of the permittivity on temperature (Fogaling et al., 2013; Turnbull and Melcher, 1969; Yoshikawa et al., 2013); the electroconductive mechanism is associated with the dependence of the conductivity on temperature (Il’in and Kartavykh, 2018; Il’in and Smorodin, 2007; Kartavykh et al., 2015; Smorodin and Verlade, 2000).

The competition of thermogravitational and electroconductive mechanisms of convection occurrence in the poorly conducting fluid placed in the alternating electric field is investigated in this paper. Unlike previous articles, in which the wave number was constant, the wave number was varied in this paper and the convection threshold at a
fixed set of parameters (modulation amplitude and frequency) corresponds to the minima of marginal stability curves of electroconvection. According to the linear theory, the disturbances, whose characteristic size corresponds to wave numbers taken in the minima of the marginal stability curves, are the most dangerous. Consequently, such extension of our previous study is important because nonlinear evolution of the most dangerous disturbances is investigated.

2. PROBLEM STATEMENT

The flat horizontal layer of the poorly conducting viscous incompressible fluid is considered. The layer is placed in the vertical alternating electric field $\vec{E}$ and the gravity field $\vec{g}$ and heated from above (Fig. 1).

The problem is considered in the electrohydrodynamics approximation, where all of the magnetic effects are negligible compared to the electric ones. Boussinesq approximation (Landau and Lifshitz, 1987), which means that the density differences are applied only in terms related to body forces, is also used. The main equation system contains the Navier-Stokes equation, the heat equation, and the charge conservation equation:
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FIG. 1: The coordinate system and geometry of the considered problem

\[
\begin{align*}
\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) &= -\nabla p + \rho \vec{g} + \eta \Delta \vec{v} + \rho_e \vec{E}, \\
\frac{\partial \theta}{\partial t} + (\vec{v} \cdot \nabla) \theta &= \chi \Delta \theta, \\
\frac{\partial \rho_e}{\partial t} + \text{div}(\sigma \vec{E}) + (\vec{v} \cdot \nabla) \rho_e &= 0,
\end{align*}
\]

where \(\vec{v}, p, \text{ and } \theta\) are two-dimensional velocity, pressure, and temperature fields; \(\eta\) is the dynamic viscosity of the fluid; \(\rho\) is the density of the fluid; \(\chi\) is the thermal diffusivity of the fluid; \(\rho_e\) is the density of charge, dissolved in the fluid; \(\sigma\) is the conductivity of the fluid. Some additional equations need to be added to the system (1) to comply with the condition of determinacy. These are

\[
\text{div}(\varepsilon \vec{E}) = \rho_e, \quad \text{div} \vec{v} = 0, \quad \vec{E} = -\nabla \phi,
\]

where \(\varepsilon\) is the permittivity of the fluid and \(\phi\) is the electric potential.

The boundary conditions are

\[
\begin{align*}
z &= 0 : & \vec{v} &= 0, & \theta &= \Theta, & \phi &= \hat{U} \cos(\omega t), \\
z &= h : & \vec{v} &= 0, & \theta &= 0, & \phi &= 0,
\end{align*}
\]

where \(\Theta\) is the temperature difference between vertical bounds; \(\omega\) and \(\hat{U}\) are the frequency and the amplitude of the external electric field.

There is no Joule heating term in the heat equation from system (1) because it can be neglected for poorly conducting fluid with conductivity \(\sigma \sim 10^{-9} - 10^{-11} \text{Ω}^{-1} \cdot \text{m}^{-1}\) and the scalar square of electric field density \(|\vec{E}|^2 < 10^5 \text{V/m}\) (Bologa et al., 1977; Smorodin and Verlade, 2000; Yin et al., 2006).

The problem is considered in the following approximation: density and conductivity of the fluid are linearly dependent on temperature (Bologa et al., 1977; Landau and Lifshitz, 1987) according to the following laws:

\[
\rho = \rho_0 (1 - \beta T) \quad \text{and} \quad \sigma = \sigma_0 (1 + \beta_0 T),
\]

where \(\rho_0\) and \(\sigma_0\) are the density and conductivity values at average temperature, \(\beta\) and \(\beta_0\) are some positive coefficients. Since the convection occurs due to the spatial heterogeneity of density and conductivity of the fluid, the coefficient \(\beta \sim 10^{-3} \text{°C}^{-1}\), \(\beta_0 \sim 10^{-2} \text{°C}^{-1}\), so in common (temperature range is \(0 \div 100 \text{°C}\)) conditions with moderate \((\sim 10 \text{°C})\) heating \(\beta \Theta < 1, \beta_0 \Theta < 1\). Theoretical investigations of electrothermal convection in static fields using this condition fit the experiment well (Zhdanov et al., 2000), so convection occurs due to the thermogravitational and the electroconductive mechanisms.

The variables are nondimensionalized this way:

\[
[r] = \frac{\rho_0 h^2}{\eta}, \quad \varphi = \hat{U}, \quad [\vec{v}] = \frac{\chi}{\eta}, \quad [\rho] = h,
\]

\[
[\Theta] = \Theta, \quad [E] = \frac{\hat{U}}{h}, \quad [\rho_e] = \frac{\eta \chi}{h^2}, \quad [\rho_e] = \frac{\varepsilon \hat{U}}{h^2}.
\]

Then the fields \(\vec{v}, \rho, \rho_e,\) and \(T\) are replaced by sums of average values and deviations according to the perturbation theory; the stream function \(\psi\) is entered and determined as \(v_x = -\partial \psi / \partial z, v_z = \partial \psi / \partial x\). After all of these manipulations the system (1) becomes
The time derivatives are indicated hereinafter by the dot (Il’in and Smorodin, 2007): to the description of convection in periodic conditions (5). Despite the small amount of basic functions, this approach (Ahlers et al., 1985; Finucane and Kelly, approximations must respond to the boundary conditions (5):

$$\text{boundary conditions of the system (4) are}$$

$$z = 0, \quad \theta = 0, \quad \psi = \psi'' = 0. \quad (5)$$

The system (4) contains the following nondimensional parameters: the Rayleigh number \(Ra\), the Prandtl number \(Pr\), the electric Prandtl number \(Pr_e\), and the electric analog of the Rayleigh number \(Ra_e\). These are determined as

$$Ra = \frac{\rho_0 g \beta \Theta h^3}{\chi \eta}, \quad Ra_e = \frac{\varepsilon U^2 \beta_0 \Theta}{\chi \eta}, \quad Pr = \frac{\eta}{\chi \rho_0}, \quad Pr_e = \frac{\varepsilon \eta}{\hbar^2 \sigma_0 \rho_0}. \quad (6)$$

Galerkin’s method with a small amount of basic functions is used to solve the equation system (4) with boundary conditions (5). Despite the small amount of basic functions, this approach (Ahlers et al., 1985; Finucane and Kelly, 1976) to the description of convection in periodic fields fits the experimental data well (Finucane and Kelly, 1976). The stream function, the temperature, and the charge density are approximated by the following relations. These approximations must respond to the boundary conditions (5):

$$\psi = (A_1(t) \sin \pi z + A_2(t) \sin 2\pi z) \sin \pi k x, \quad (7)$$

$$\theta = (B_1(t) \sin \pi z + B_2(t) \sin 2\pi z) \cos \pi k x + C(t) \sin 2\pi z, \quad (7)$$

$$\rho_e = (D_1(t) \cos \pi z + D_2(t) \cos 2\pi z) \cos \pi k x + E(t) \cos 2\pi z. \quad (7)$$

Here \(k\) is the wave number, that characterizes the periodicity of perturbations in the plane of the layer; \(A_1, A_2, B_1, B_2, C, D_1, D_2, E\) are the amplitudes of different spatial modes.

The relations (7) are applied to the system (4), then the system (4) is orthogonalized according to the Galerkin method and, finally, the system of eight ordinary differential equations for the space harmonic amplitudes appears (the time derivatives are indicated hereinafter by the dot) (Il’in and Smorodin, 2007):

$$\dot{X} = Pr(-X + r Y - \epsilon T \cos \omega t),$$
$$\dot{Y} = -Y + X + X Z,$$
$$\dot{Z} = -b Z - X Y,$$
$$\dot{V} = Pr(-d \cdot V + (r W + \epsilon S \cos \omega t)/d),$$
$$\dot{W} = -d \cdot W + V,$$
$$\dot{S} = -g S + X U - g Y \cos \omega t,$$
$$\dot{T} = -g T - g W \cos \omega t,$$
$$\dot{U} = -g U - X S - 2 g Z \cos \omega t. \quad (8)$$

In the system (8) \(X\) and \(V\) are the stream function amplitudes; \(Y, Z, \) and \(W\) are the temperature amplitudes; \(S, T,\) and \(U\) are the charge density amplitudes. The system (8) contains the following parameters:

$$r = \frac{Ra}{Ra_0}, \quad e = \frac{Ra_e}{Ra_0}, \quad Ra_0 = \frac{\pi^4(1 + k^2)^3}{k^2}, \quad Ra_e = \frac{3\pi^4(1 + k^2)^3}{8k^2}, \quad (9)$$

$$b = \frac{4}{1 + k^2}, \quad d = \frac{4 + k^2}{1 + k^2}, \quad g = \frac{Pr}{\pi^2(1 + k^2)Pr_e}, \quad$$

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where \( r \) and \( e \) are normalized thermal and electric Rayleigh numbers, \( \text{Ra}_0 \) and \( \text{Ra}_{\sigma_0} \) are critical numbers from which begins thermogravitational or electroconductive convection accordingly; \( k \) is included in relations (7) wave number, which determines the size of the convection cells; \( b, d, g \) are geometric parameters.

### 3. APPLICATION OF FLOQUET THEORY TO INVESTIGATE LINEAR INSTABILITY OF THE ELECTROCONVECTION LOW-MODE MODEL

Using the small perturbation theory, the system (8) can be represented in the following form:

\[
\dot{x}(t) = \hat{A}(t)x(t),
\]

where \( x(t) = \{X(t), Y(t), Z(t), V(t), W(t), S(t), T(t), U(t)\} \). Then the system matrix \( \hat{A}(t) \) is in the linear approximation

\[
\hat{A}(t) = \begin{pmatrix}
-Pr & rPr & 0 & 0 & 0 & 0 & -ePr \cos \omega t & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -d & \frac{rPr}{d} & \frac{ePr \cos \omega t}{d} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -g \cos \omega t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -g \cos \omega t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -g \\
\end{pmatrix}.
\]

The system (10) is linear, continuous, and periodic with frequency \( \omega \) (period \( \hat{T} = 2\pi/\omega \)) and its fundamental matrix can be represented accordingly to the Floquet theory (Coddington and Levinson, 1955) in the form

\[
Q(t) = \Phi(t) \exp(\hat{B}t),
\]

where \( \Phi(t) \) is the continuous periodic matrix with the same period as the matrix \( \hat{A}(t) \), \( \hat{B} \) is a constant matrix, so

\[
Q(t + T) = \Phi(t + \hat{T}) \exp(\hat{B}(t + \hat{T}) = \Phi(t) \exp(\hat{B}t) \exp(\hat{B}\hat{T}) = Q(t) \exp(\hat{B}\hat{T}) \equiv Q(t + \hat{T}),
\]

then

\[
\exp(\hat{B}\hat{T}) = Q(\hat{T}).
\]

The constant matrix \( Q(\hat{T}) \) is called the monodromy matrix. The eigenvalues \( \rho_j \) (they can be complex) of the monodromy matrix \( Q(\hat{T}) \) are called the characteristic multipliers for the system (10).

The fact that linear instability of the system (10) is determined by the characteristic multipliers of the monodromy matrix \( Q(\hat{T}) \) can be proved according to the Lyapunov-Perron transformation, so if all of the characteristic multipliers \( \rho_j \) of the monodromy matrix \( Q(\hat{T}) \) fulfill the condition \( |\rho| \leq 1 \), the system (10) is stable. If the multipliers fulfill the condition \( |\rho| < 1 \), the system (10) is asymptotically stable. In all other cases the system (10) is unstable.

The characteristic multipliers can also be represented as \( \rho_j = \exp(\lambda_j \hat{T}) \), where \( \lambda_j = \Re(\lambda_j) + i\Im(\lambda_j) \) are the complex Lyapunov exponents of the system (10), so the characteristic multipliers are

\[
\rho_j = \exp((\Re(\lambda_j) + i\Im(\lambda_j))\hat{T}) = |\rho| \exp(i\Im(\lambda_j)\hat{T}).
\]

Lyapunov exponents can be ordered this way: \( \Re(\lambda_1) > \Re(\lambda_2) > \ldots > \Re(\lambda_j) \). The condition \( \Re(\lambda_1) = 0 \) \((|\rho_1| = 1)\) determines the area of periodic solutions (neutral perturbations) in the space of parameters \( k, e, \omega \). The imaginary part of the Lyapunov exponent determines the type of oscillations of the system (10). If \( \Im(\lambda_1) \) is equal to the external influence frequency \( \omega \), then the characteristic multiplier \( \rho_1 \) equals 1 and this corresponds to stable synchronous (with the same period as external influence) oscillation regimes. If \( \Im(\lambda_1) \) is equal to \( \omega/2 \), then the characteristic multiplier \( \rho_1 \) equals \(-1\) and this corresponds to stable subharmonic (with the period twice as large
as the external influence period) oscillation regimes. If the characteristic multiplier is complex, but equals 1 in the absolute value, it complies with stable quasiperiodic oscillation regimes.

The convection in the investigated physical system begins at the moment when at least one of multipliers of the system monodromy matrix becomes equal to 1 in the absolute value. The monodromy matrix can be calculated by numerically integrating the system (10) on its period $\hat{T}$.

4. THE INVESTIGATION RESULTS OF LINEAR INSTABILITY OF THE ELECTROCONVECTION LOW-MODE MODEL

The equilibrium stability of poorly conducting fluid placed in the capacitor and the evolution of flows were investigated in alternating field for poorly conducting fluids, such as capacitor and transformer oil, and Mazola corn oil with the set of parameters $Pr = 400$, $Pr_e = 30$ (Zhdanov et al., 2000). The considered fluid is heated from above ($\Theta < 0$). In this case normalized Rayleigh numbers $r$ and $e$ are negative according to the definitions (6, 9), $r = -1$; the normalized Rayleigh number $e$ will be considered in its absolute value for convenience. The marginal stability curves for different external electric field periods $1/\nu$ ($\nu$ is linear frequency) in coordinates “wave number $k$ – nondimensional electric parameter $|e|$” are depicted in Fig. 2.

The critical values of the nondimensional electric parameter $|e_{\text{min}}|$ and the wave number $k_c$ (i.e., values at which convection of the fluid starts) are the extreme points of the marginal stability curves. For example, $|e_{\text{min}}|(1/\nu = 2.22) = 91.7$ and $k_c(1/\nu = 2.22) = 1.47$. The characteristic multipliers are complex in the regions of the global minimum of the marginal stability curves. This means that electroconvection occurs in the form of quasiperiodic oscillations at the critical wave number value. There are local extrema on marginal stability curves. The characteristic multipliers are real in the regions of these “bags” and this corresponds to the electroconvection synchronous regimes. Quasiperiodic oscillations regions are marked in Fig. 2 with solid lines; the regions corresponding to synchronous oscillations are marked with dashed lines.

The critical values $|e_{\text{min}}|$ and $k_c$ for external electric field periods $1/\nu \in [2, 5]$ were found analogically (Fig. 3).

![FIG. 2: The marginal stability curves family for different external electric field periods in coordinates “wave number $k$ – nondimensional electric parameter $|e|$” at the following set of parameters: $Pr = 400$, $Pr_e = 30$, $r = -1$. The solid line corresponds to quasiperiodic regimes; dashed line corresponds to synchronous regimes.](image-url)
FIG. 3: The critical value of the nondimensional electric parameter $|e_{\text{min}}|$ (a) and the critical value of the wave number $k_c$ (b) vs external electric field period $1/\nu$ at the following set of parameters: $Pr = 400, Pr_e = 30, r = -1$. Cuts numbering matches curves numbering in Fig. 2.

5. NONLINEAR REGIMES OF CONVECTION IN THE ELECTROCONVECTION LOW-MODE MODEL

The nonlinear system (8) is integrated using the Runge-Kutta-Merson method with fourth-order accuracy. Time averaged nondimensional heat flow on the bound of the fluid [the Nusselt number (Gershuni and Zhukovitskii, 1976)] is calculated to analyze the intensity of heat transfer through the capacitor this way:

$$\text{Nu} = 1 - \frac{2}{t_{\text{end}}} \int_0^{t_{\text{end}}} Z(t) dt, \quad \text{Nu} = \frac{q h}{\kappa \Theta},$$

(15)

where $Z(t)$ is temperature amplitude; $q$ is the heat flow density; $\kappa$ is the thermal conductivity. The Nusselt numbers were averaged by wide time interval $t_{\text{end}} = N \cdot (1/\nu)$, $(N > 100)$. The case $\text{Nu} = 1$ corresponds to the process of molecular heat transfer; the excess of Nusselt number above one $\text{Nu} > 1$ corresponds to the convection appearance.

At each step of numerical integration values received on the previous step will be used as the initial values. This method allows one to consider the evolution of the system in the case of continuous change of the control parameters such as non-dimensional electric parameter $e$.

Nusselt number $\text{Nu}$ as a function of the nondimensional electric parameter absolute value $|e|$ at the external field frequency 0.45 (period $1/\nu = 2.22$) is shown in Fig. 4. In the case of increasing $|e|$, the convection starts at the point B ($|e| = 91.7$) in the form of the quasiperiodic oscillations; i.e., the Fourier spectrum of amplitude $X$ contains two frequencies, which are aliquant to the external field frequency; all other frequencies of the spectrum are their linear combinations (Berge et al., 1986). The quasiperiodic oscillations are continuing to point C ($|e| = 102.5$) where chaos appears; i.e., the Fourier spectrum becomes continuous. Chaos exists on the segment CF, except from the heat flow leap region DE (from $|e| = 105.7$ to $|e| = 106.6$), where the Fourier spectrum of amplitude $X$ contains the frequency
FIG. 4: Nusselt number \( \text{Nu} \) as a function of the nondimensional electric parameter \( |e| \) at the external field frequency 0.45 and the following set of parameters: \( \text{Pr}_r = 400, \text{Pr}_e = 30, r = -1 \). The solid line corresponds to regular regimes; chaos regions are plotted with dots; arrows show direction of electric parameter \( |e| \) change (increase or decrease of \( |e| \)).

which is three times less than the external field frequency. This is called the periodic window. The nondimensional heat flow intensively increases at the point F (\( |e| = 108.2 \)), and the system starts to oscillate in the synchronous regime at point G; i.e., the Fourier spectrum of amplitude \( X \) contains only those frequencies, which are aliquot to the external field frequency.

The nondimensional electric parameter \( |e| \) can be decreased from the values, which lie to the right of point G (Fig. 4); in this case the system will evolve in a different way, so hysteresis exists in the considered system. There is a synchronous regime on the top branch of the hysteresis (segment GH from \( |e| = 108.5 \) to \( |e| = 70.6 \)). Then, the Nusselt number intensively decreases at point H and the system returns to the unexcited state at point A.

6. CONCLUSIONS

In the framework of the electroconvection low-mode model we have investigated the behavior of the poorly conducting fluid heated from above in the alternating electric field. Based on the linear theory, using the analysis of marginal stability curves, the occurrence of equilibrium instability has been studied. We have established that the instability occurs due to the quasiperiodic and synchronous vibrational modes. The dependences of the critical wave numbers and dimensionless electric parameter on the field frequency have been obtained. It has been shown that the wave number monotonically increases and the absolute value of the electric parameter decreases with increasing of the inverse frequency.

The critical wave number obtained in linear theory has been used to study the nonlinear evolution of the system. The dependence of Nusselt number on the electric parameter has been analyzed. It has been found that the fluid motion is associated with various types of vibrational modes. We have revealed that the transition to chaos occurs through the quasiperiodicity. The coexistence regions of oscillation regimes with different values of heat flow were discovered. We have shown that synchronous oscillations can arise in the system under consideration that compete with quasiperiodic, chaotic oscillations, or an unexcited state.
REFERENCES


